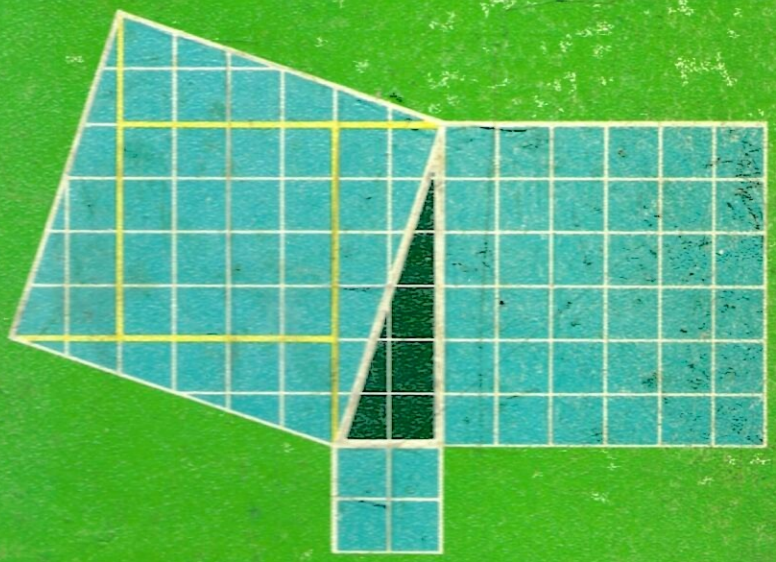
SCHOOL MATHEMATICS

I



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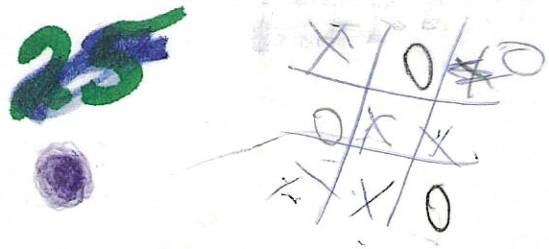

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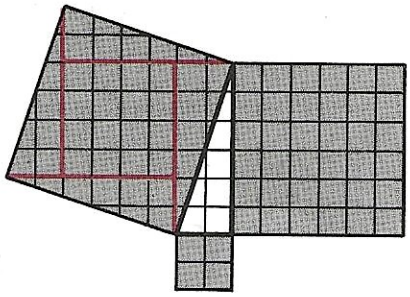

I

SCHOOL MATHEMATICS

ROBERT E. EICHOLZ
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CHARLES R. FLEENOR

CALIFORNIA STATE SERIES
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Chapter Openings

Chapter frontispiece quotations are taken from Lewis Carroll's *Alice in Wonderland* and *Through the Looking Glass*. Frontispiece illustrations are adapted from John Tenniel's drawings for Lewis Carroll's books.

Lewis Carroll, whose actual name was Charles Dodgson, was an accomplished mathematician as well as a writer, and his fictional works (such as the two named above) frequently use mathematical ideas in a humorous and paradoxical manner. Consequently, the chapter frontispiece quotations are chosen not only for their humor, but also for their relevance to the material in the chapter.

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To the Teacher

In School Mathematics I *and* II, *as in the* Elementary School Mathematics *series, there is a twofold purpose. The entire program is committed to the encouragement of an enthusiastic curiosity about the basic concepts and structure of mathematics and to the development of proficiency in mathematical skills, including an understanding of their application to our contemporary, quantitative world.*

Each text emphasizes the discovery approach to learning, often utilizing a nonverbal presentation of mathematics through diagrams, charts, and sequences of discovery activities. Flexibility is an important feature of each text. The unusual variety of standard materials, as well as new topics, enrichment materials, and exercises graduated in difficulty, provides an appeal for students of widely divergent abilities and interests.

Above all, the authors intend that School Mathematics I *and* II *provide the atmosphere and the tools for an exciting adventure in mathematics, developing the student's ability to move ahead and attack new problems with confidence.*

To the Student

You will notice from the beginning that we are very much concerned in this text with the "why" of things. Much stress is placed upon understanding concepts and discovering new mathematical ideas. While it is important that you develop skill in mathematics, you will also need to know why various procedures work, as well as when and where you should use them. Perhaps the most important thing you can gain from this text is the ability to attack new problems and think for yourself.

Study carefully. As you study, keep pencil and paper at hand. Work out the examples to make sure you understand the ideas. Ask questions in class when you do not thoroughly understand what is being discussed. There are many challenging ideas in this book that you will enjoy mastering. Your real reward will come when you know that you have learned and understood an important new idea.

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CONTINUES
FROM PART 1

". . . You see the earth takes twenty-four hours to turn round on its axis——"

"Talking of axes," said the Duchess, "Chop off her head!"

Alice glanced rather anxiously at the cook, to see if she meant to take the hint; but the cook was busily stirring the soup, and seemed not to be listening, so she went on again: "Twenty-four hours, I think; *or is it twelve? I ——"*

"Oh, don't bother me," *said the Duchess. "I never could abide figures!"*

8

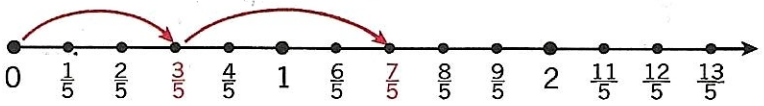
Computing With Rational Numbers

8-1 THE NUMBER LINE

The number line can help us think about addition and subtraction of rational numbers.

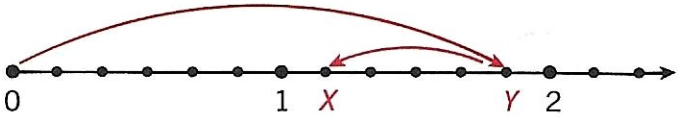
Discussion Exercises

1. The number line is used to illustrate the addition fact $\frac{3}{5} + \square = \frac{7}{5}$. What fraction is covered by the $\square$?



2. The number line is scaled in sixths.

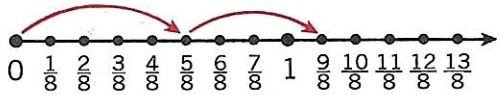
[a] What rational number does the point labeled *Y* suggest?
[b] What rational number does the point labeled *X* suggest?
[c] What subtraction equation is illustrated?

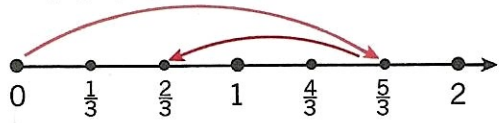


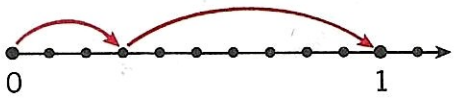
3. Tell how addition is always pictured on the number line. How does this differ from the way subtraction is pictured?

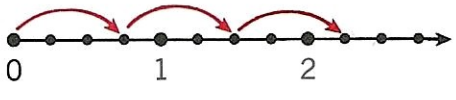
Exercises

1. What addition or subtraction equation is suggested by each number line?

[a] 

[b] 

[c] 

[d] 

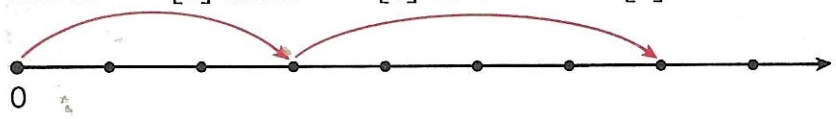
2. Find the sum or difference.

[a] $\frac{3}{4}+\frac{2}{4}$ [c] $\frac{16}{5}-\frac{8}{5}$ [e] $\frac{5}{6}+\frac{3}{6}$ [g] $\frac{3}{8}+\frac{5}{8}$ [i] $\frac{43}{100}+\frac{21}{100}$ ★[k] $(\frac{5}{3}-\frac{2}{3})-\frac{1}{3}$

[b] $\frac{7}{8}-\frac{3}{8}$ [d] $\frac{7}{12}+\frac{9}{12}$ [f] $\frac{21}{9}-\frac{4}{9}$ [h] $\frac{21}{23}-\frac{20}{23}$ [j] $\frac{16}{25}+\frac{4}{25}$ ★[l] $\frac{5}{3}-(\frac{2}{3}-\frac{1}{3})$

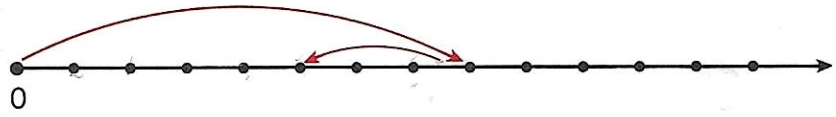
3. Give the addition fact shown by the following number line, assuming that it is scaled in

[a] halves [b] thirds [c] sevenths [d] tenths [e] twentieths

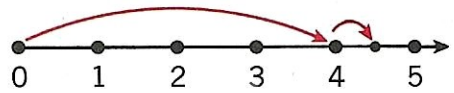


4. What subtraction fact is shown by the number line if it is scaled in

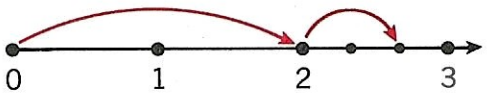
[a] twelfths? [b] tenths? [c] eighths? [d] sevenths? [e] fourths?



5. **Mixed numerals** such as $4\frac{1}{2}$ and $2\frac{3}{4}$ can be better understood with the help of the number line. Solve each equation.

[a] 

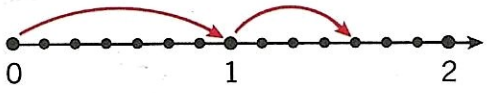
$n+\frac{1}{2}=4\frac{1}{2}$

[c] 

$2+\frac{2}{3}=n$

[b]

$2+n=2\frac{3}{4}$

[d] 

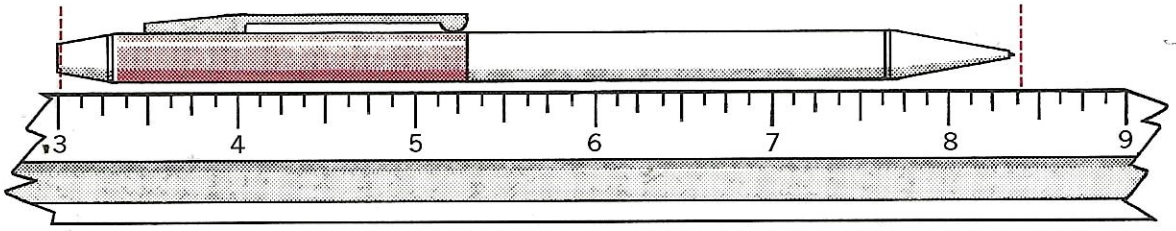
$1+n=1\frac{4}{7}$

6. Draw number lines for Parts **a** and **b**. Then give the correct mixed numeral for each sum **a** through **h**.

[a] $2 + \frac{5}{8}$ [c] $11 + \frac{5}{6}$ [e] $15 + \frac{2}{3}$ [g] $19 + \frac{13}{16}$

[b] $9 + \frac{1}{4}$ [d] $8 + \frac{7}{10}$ [f] $23 + \frac{7}{8}$ [h] $42 + \frac{19}{32}$

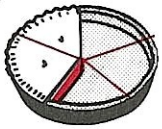
7. What is the length of the pencil?



8. If a, b, and c are whole numbers and $b \neq 0$, then $\frac{a}{b} + \frac{c}{b} =$ ▥.

SHORT STORIES

1. Ate $\frac{2}{5}$ of a pie. Then ate $\frac{1}{5}$ more. How much was eaten in all?



2. Ran $\frac{4}{9}$ mi. Walked $\frac{2}{9}$ mi. How far in all?

3. One mile home. Drove $\frac{7}{10}$ mi before running out of gas. How far to walk home?

4. Sack of gum drops. Ate $\frac{5}{12}$ of them. Gave away $\frac{3}{12}$ of them. What part was left?

5. Drove $\frac{3}{4}$ mi to store. Drove back. How far in all?

6. Homework time: English, $\frac{23}{60}$ hr; mathematics, $\frac{28}{60}$ hr; science $\frac{17}{60}$ hr. How many hours homework?

7. Rectangle: $\frac{7}{8}$ in. long, $\frac{3}{8}$ in. wide. What is the perimeter?

8. Nail A is $\frac{15}{16}$ in. long. Nail B is $\frac{9}{16}$ in. long. How much longer is A than B?

9. Distance from Town A to Town B is $\frac{4}{5}$ mi. Distance from Town B to Town C is $\frac{3}{5}$ mi. Going through B, what is the distance from A to C?

10. Height 59 in. Grew $\frac{3}{4}$ in. How tall now?

Think

Each letter names a different prime number. Find each number.

$$a \cdot b \cdot c = 1001$$

8-2 ADDING AND SUBTRACTING RATIONAL NUMBERS

When the fractions for two rational numbers have the same denominator, we may easily find the sum or difference of the numbers. If the denominators are different, we select equivalent fractions that have the same denominator.

A To find $\frac{2}{3} + \frac{3}{5}$, we think $\frac{10}{15} + \frac{9}{15} = \frac{19}{15}$.

$\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \frac{12}{18}, \ldots\right\}$ $\left\{\frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \ldots\right\}$

B To find $\frac{3}{4} - \frac{2}{7}$, we think $\frac{21}{28} - \frac{8}{28} = \frac{13}{28}$.

$\left\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{15}{20}, \frac{18}{24}, \frac{21}{28}, \ldots\right\}$ $\left\{\frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \frac{8}{28}, \ldots\right\}$

Exercises

1. Find the numerators r, s, and t.

[a] $\frac{5}{6} + \frac{3}{4} = \frac{r}{12} + \frac{s}{12} = \frac{t}{12}$

$\left\{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \ldots\right\}$ $\left\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \ldots\right\}$

[b] $\frac{1}{3} - \frac{2}{7} = \frac{r}{21} - \frac{s}{21} = \frac{t}{21}$

$\left\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}, \frac{6}{18}, \frac{7}{21}, \frac{8}{24}, \ldots\right\}$ $\left\{\frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \frac{8}{28}, \ldots\right\}$

2. Copy and complete each problem. Write a set of equivalent fractions for each number given. Then find the sum or difference. Part a is worked as a sample problem.

[a] $\frac{5}{8} \rightarrow \left\{\frac{5}{8}, \frac{10}{16}, \frac{15}{24}, \frac{20}{32}, \ldots\right\} \rightarrow \frac{15}{24}$

$+\frac{5}{12} \rightarrow \left\{\frac{5}{12}, \frac{10}{24}, \frac{15}{36}, \ldots\right\} \rightarrow +\frac{10}{24}$

$\frac{25}{24} \leftarrow \frac{25}{24}$

[b] $\frac{5}{9} \rightarrow \{\quad, \quad, \quad, \ldots\} \rightarrow$ ▮

$+\frac{1}{6} \rightarrow \{\quad, \quad, \quad, \ldots\} \rightarrow +$ ▮

▮ ← ▮

[c] $\frac{3}{5} \rightarrow \{\quad, \quad, \quad, \ldots\} \rightarrow$ ▮

$-\frac{4}{15} \rightarrow \{\quad, \quad, \quad, \ldots\} \rightarrow -$ ▮

▮ ← ▮

[d] $\frac{5}{6} \rightarrow \{\quad, \quad, \quad, \ldots\} \rightarrow$ ▮

$+\frac{7}{4} \rightarrow \{\quad, \quad, \quad, \ldots\} \rightarrow +$ ▮

▮ ← ▮

3. Give the numbers for r and s. Then give the number for t.

[a] $\left.\begin{matrix}\frac{2}{3}=\frac{r}{12}\\ \frac{3}{4}=\frac{s}{12}\end{matrix}\right\}\ \frac{2}{3}+\frac{3}{4}=\frac{t}{12}$

[b] $\left.\begin{matrix}\frac{9}{10}=\frac{r}{30}\\ \frac{3}{6}=\frac{s}{30}\end{matrix}\right\}\ \frac{9}{10}+\frac{3}{6}=\frac{t}{30}$

[c] $\left.\begin{matrix}\frac{5}{9}=\frac{r}{27}\\ \frac{4}{27}=\frac{s}{27}\end{matrix}\right\}\ \frac{5}{9}-\frac{4}{27}=\frac{t}{27}$

[d] $\left.\begin{matrix}\frac{3}{10}=\frac{r}{100}\\ \frac{7}{100}=\frac{s}{100}\end{matrix}\right\}\ \frac{3}{10}+\frac{7}{100}=\frac{t}{100}$

[e] $\left.\begin{matrix}\frac{7}{5}=\frac{r}{45}\\ \frac{7}{9}=\frac{s}{45}\end{matrix}\right\}\ \frac{7}{5}+\frac{7}{9}=\frac{t}{45}$

[f] $\left.\begin{matrix}\frac{5}{8}=\frac{r}{24}\\ \frac{5}{12}=\frac{s}{24}\end{matrix}\right\}\ \frac{5}{8}-\frac{5}{12}=\frac{t}{24}$

[g] $\left.\begin{matrix}\frac{4}{9}=\frac{r}{144}\\ \frac{5}{16}=\frac{s}{144}\end{matrix}\right\}\ \frac{4}{9}-\frac{5}{16}=\frac{t}{144}$

★[h] $\left.\begin{matrix}\frac{5}{23}=\frac{r}{943}\\ \frac{3}{41}=\frac{s}{943}\end{matrix}\right\}\ \frac{5}{23}+\frac{3}{41}=\frac{t}{943}$

★[i] $\left.\begin{matrix}\frac{13}{15}=\frac{r}{120}\\ \frac{9}{40}=\frac{s}{120}\end{matrix}\right\}\ \frac{13}{15}+\frac{9}{40}=\frac{t}{120}$

4. Give the numbers for x, y, and z.

[a] $\left.\begin{matrix}\frac{3}{5}=\frac{x}{30}\\ \frac{2}{15}=\frac{y}{30}\end{matrix}\right\}\ \frac{3}{5}+\frac{2}{15}=\frac{z}{30}$

[b] $\left.\begin{matrix}\frac{3}{5}=\frac{x}{15}\\ \frac{2}{15}=\frac{y}{15}\end{matrix}\right\}\ \frac{3}{5}+\frac{2}{15}=\frac{z}{15}$

[c] $\left.\begin{matrix}\frac{5}{12}=\frac{x}{72}\\ \frac{7}{18}=\frac{y}{72}\end{matrix}\right\}\ \frac{5}{12}+\frac{7}{18}=\frac{z}{72}$

[d] $\left.\begin{matrix}\frac{5}{12}=\frac{x}{36}\\ \frac{7}{18}=\frac{y}{36}\end{matrix}\right\}\ \frac{5}{12}+\frac{7}{18}=\frac{z}{36}$

[e] Are the answers for Parts **a** and **b** different? Why?
[f] Do the answers for Parts **c** and **d** name the same number? Why?

5. Find the sum or difference.

[a] $\frac{1}{6}+\frac{1}{4}$ [b] $\frac{11}{8}-\frac{2}{3}$ [c] $\frac{7}{10}+\frac{1}{4}$ [d] $\frac{3}{10}+\frac{4}{15}$ [e] $\frac{15}{16}-\frac{3}{4}$ [f] $\frac{4}{5}-\frac{3}{4}$ [g] $\frac{5}{8}+\frac{1}{3}$ [h] $\frac{7}{13}+\frac{1}{3}$ [i] $\frac{11}{85}+\frac{4}{51}$ [j] $\frac{17}{26}-\frac{11}{65}$

Think

Ⓐ $15\overline{)n}$, with remainder 13 (R13)

Ⓑ $18\overline{)n}$, R1

The number n in Problem A is the same as in Problem B. The quotient in Problem A is the same as in Problem B. Find n and the missing quotient.

3. [a] Is 10 a multiple of 6?
[b] Is $2 \cdot 10$ a multiple of 6?
[c] Is $3 \cdot 10$ a multiple of 6?
[d] What is the LCM of 6 and 10?
[e] What is the LCD for $\frac{1}{6}$ and $\frac{3}{10}$?

4. By finding the LCM of the denominators, find the LCD for the pair of fractions.

[a] $\frac{2}{3}, \frac{7}{9}$ [b] $\frac{1}{6}, \frac{1}{8}$ [c] $\frac{2}{15}, \frac{3}{10}$ [d] $\frac{5}{8}, \frac{1}{12}$ [e] $\frac{1}{4}, \frac{1}{9}$ [f] $\frac{2}{7}, \frac{5}{8}$ [g] $\frac{1}{3}, \frac{4}{5}$ [h] $\frac{3}{10}, \frac{7}{100}$ [i] $\frac{7}{12}, \frac{9}{16}$ [j] $\frac{1}{2}, \frac{3}{11}$ [k] $\frac{5}{9}, \frac{1}{12}$ [l] $\frac{11}{16}, \frac{5}{6}$

5. [a] Give two common multiples of 6, 8, and 10.
[b] What is the LCM of 6, 8, and 10?
[c] What is the LCD for $\frac{1}{6}$, $\frac{1}{8}$, and $\frac{1}{10}$?

6. [a] Give two common multiples of 16 and 40.
[b] Give the LCM of these two numbers.
[c] What is the LCD for $\frac{1}{16}$ and $\frac{1}{40}$?
[d] Find the sum $\frac{5}{16} + \frac{7}{40}$.

7. [a] Find the LCM of 27 and 45.
[b] What is the LCD for $\frac{7}{27}$ and $\frac{11}{45}$?
[c] Find $\frac{7}{27} + \frac{11}{45}$.
[d] Find $\frac{7}{27} - \frac{11}{45}$.

8-4 METHODS FOR ADDING AND SUBTRACTING RATIONAL NUMBERS

In simple addition problems, practice finding the LCD mentally.

Problem	$\frac{3}{4} + \frac{1}{6}$	$\frac{5}{8} - \frac{1}{12}$	$\frac{13}{14} - \frac{25}{42}$
We think	LCD is 12.	LCD is 24.	LCD is 42.
We write	$\frac{3}{4} = \frac{9}{12}$ $+\frac{1}{6} = \frac{2}{12}$ $\frac{11}{12}$	$\frac{5}{8} = \frac{15}{24}$ $-\frac{1}{12} = \frac{2}{24}$ $\frac{13}{24}$	$\frac{13}{14} = \frac{39}{42}$ $-\frac{25}{42} = \frac{25}{42}$ $\frac{14}{42} = \frac{1}{3}$ (reduced to lowest terms)

Exercises

1. Copy each problem and give the missing numerators. Then give the sum or difference.

[a] $\frac{2}{3} = \frac{\square}{6}$ $+\frac{1}{2} = \frac{\square}{6}$

[b] $\frac{5}{4} = \frac{\square}{12}$ $-\frac{5}{6} = -\frac{\square}{12}$

[c] $\frac{2}{9} = \frac{\square}{18}$ $+\frac{5}{6} = \frac{\square}{18}$

[d] $\frac{3}{10} = \frac{\square}{40}$ $+\frac{3}{8} = \frac{\square}{40}$

[e] $\frac{5}{6} = \frac{\square}{12}$ $-\frac{1}{4} = -\frac{\square}{12}$

2. Find the sum or difference. Reduce your answer to lowest terms.

[a] $\frac{2}{3} + \frac{3}{4}$

[b] $\frac{3}{4} + \frac{3}{8}$

[c] $\frac{11}{12} + \frac{1}{4}$

[d] $\frac{7}{8} - \frac{5}{6}$

[e] $\frac{11}{20} - \frac{3}{8}$

[f] $\frac{15}{16} - \frac{3}{4}$

[g] $\frac{5}{6} - \frac{1}{3}$

[h] $\frac{4}{5} + \frac{1}{6}$

[i] $\frac{3}{2} - \frac{3}{4}$

[j] $\frac{9}{10} - \frac{1}{3}$

[k] $\frac{77}{100} - \frac{7}{10}$

[l] $\frac{3}{5} + \frac{1}{2}$

★[m] $\frac{12}{20} + \frac{14}{16}$

★[n] $\frac{18}{27} - \frac{32}{64}$

★[o] $\frac{18}{28} + \frac{4}{56}$

3. For each problem, rename two of the three rational numbers so that the fractions have the same denominators. Find the sum of the three numbers.

[a] $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$

[b] $\frac{1}{4}, \frac{3}{8}, \frac{5}{16}$

[c] $\frac{1}{2}, \frac{3}{4}, \frac{5}{16}$

[d] $\frac{3}{2}, \frac{2}{3}, \frac{5}{6}$

[e] $\frac{7}{2}, \frac{6}{5}, \frac{3}{10}$

[f] $\frac{1}{3}, \frac{4}{5}, \frac{7}{15}$

[g] $\frac{3}{5}, \frac{1}{6}, \frac{7}{30}$

[h] $\frac{5}{8}, \frac{6}{7}, \frac{3}{56}$

[i] $\frac{4}{7}, \frac{7}{10}, \frac{11}{70}$

[j] $\frac{5}{4}, \frac{11}{25}, \frac{71}{100}$

4. You can find the LCD for two fractions by thinking about the prime factorization of their denominators.

$\frac{9}{30} + \frac{13}{42}$

Prime factorization of 30 = $2 \cdot 3 \cdot 5$

Prime factorization of 42 = $2 \cdot 3 \cdot 7$

LCM of 30 and 42: $2 \cdot 3 \cdot 5 \cdot 7 = 210$

[a] Use the prime factorizations to help find the missing number for each ▒.

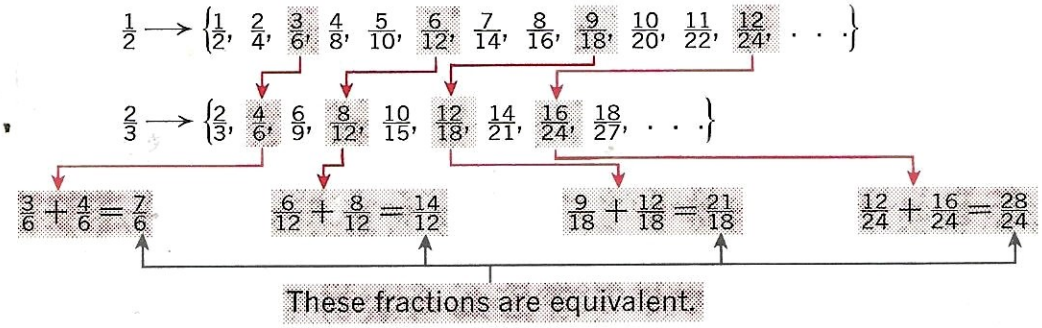
1. $2 \cdot 3 \cdot 5 \cdot ▒ = 210$
2. $30 \cdot ▒ = 210$
3. $2 \cdot 3 \cdot 7 \cdot ▒ = 210$
4. $42 \cdot ▒ = 210$

[b] What is the number for r?
[c] What is the number for s?
[d] What is the number for t?

$\left.\begin{array}{l} \frac{9}{30} = \frac{r}{210} \\ \frac{13}{42} = \frac{s}{210} \end{array}\right\} \frac{9}{30} + \frac{13}{42} = \frac{t}{210}$

8-3 LEAST COMMON DENOMINATOR (LCD)

You can find the sum of two rational numbers by choosing for these numbers fractions that have the same denominator. You can make different choices for a common denominator, but you will always get the same rational number for the sum. The fractions for the sums in the examples below are equivalent fractions because they name the same rational number. Consider the sum $\frac{1}{2} + \frac{2}{3}$.



From this example you can see that

$$\frac{1}{2} + \frac{2}{3} = \frac{7}{6} = \frac{14}{12} = \frac{17}{18} = \frac{28}{24} \cdots$$

Although you may use many different pairs of fractions to find a sum, your work will be easier if you choose pairs with the smallest possible common denominator. This denominator is called the **least common denominator** (LCD) for the two fractions given. In the example above, 6 was the LCD for $\frac{1}{2}$ and $\frac{2}{3}$. Notice that 6 is the least common multiple (LCM) of the denominators 2 and 3.

Exercises

1. Multiples of 6 → {6, 12, 18, 24, 30, 36, 42, 48, . . .}
Multiples of 8 → {8, 16, 24, 32, 40, 48, 56, 64, . . .}

[**a**] Is 36 a multiple of 6? A multiple of 8?
[**b**] Is 48 a multiple of both 6 and 8?
[**c**] What is the LCM of 6 and 8?
[**d**] What is the LCD for $\frac{1}{6}$ and $\frac{1}{8}$?

2. [**a**] List the multiples of 8 up to 80.
[**b**] List the multiples of 14 up to 140.
[**c**] From the lists of Parts **a** and **b**, select the LCM of 8 and 14.
[**d**] What is the LCD for $\frac{3}{8}$ and $\frac{5}{14}$?

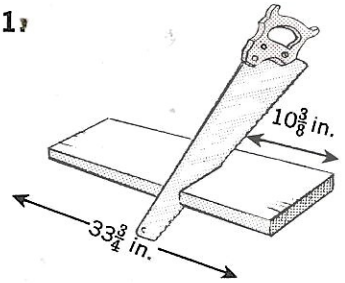
5. Use the method of Exercise 4 to find the sums.

[a] $\frac{1}{35} + \frac{2}{55}$ [b] $\frac{2}{15} + \frac{2}{21}$ [c] $\frac{5}{14} + \frac{3}{22}$ [d] $\frac{7}{24} + \frac{4}{9}$ [e] $\frac{3}{22} + \frac{5}{26}$ [f] $\frac{8}{105} + \frac{5}{63}$ [g] $\frac{1}{3} + \frac{1}{4} + \frac{1}{6}$ [h] $\frac{2}{5} + \frac{1}{6} + \frac{7}{30}$ ★[i] $\frac{5}{30} + \frac{7}{42} + \frac{9}{54}$ ★[j] $\frac{7}{35} + \frac{3}{18} + \frac{7}{105}$ ★[k] $\frac{7}{9} + \frac{5}{24} + \frac{11}{36}$ ★[l] $\frac{3}{8} + \frac{5}{12} + \frac{7}{10}$

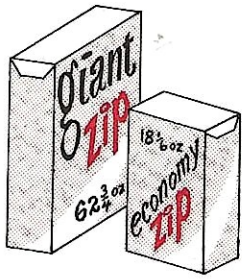
WORDLESS PROBLEMS

For each picture, write and solve a problem using rational numbers.

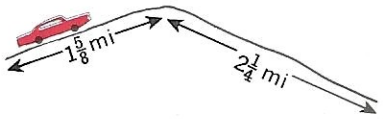
1.



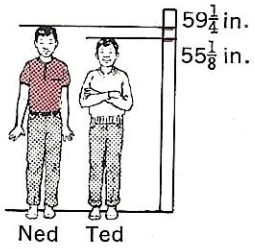
2.



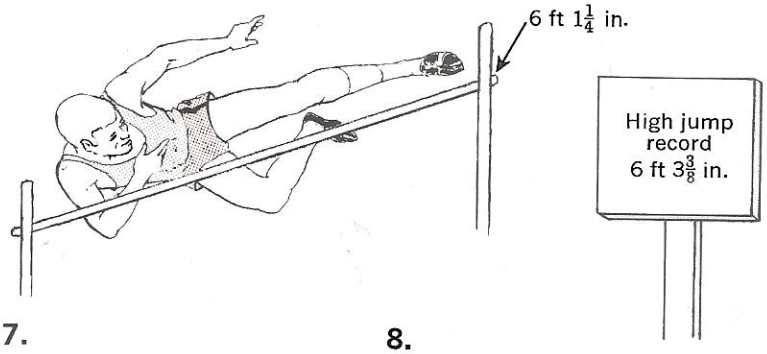
3.



4.



5.



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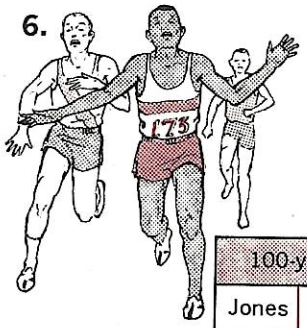
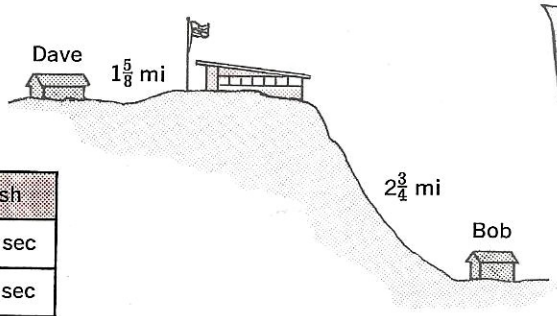
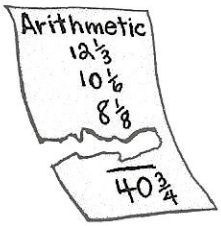


100-yd dash	
Jones	$9\frac{9}{10}$ sec
Brown	$10\frac{1}{10}$ sec
White	$10\frac{5}{10}$ sec

7.



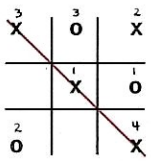
8.



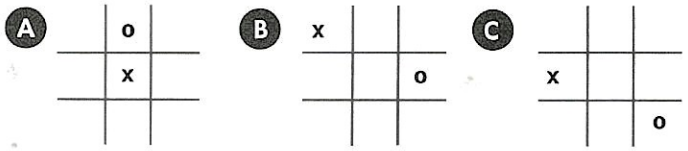
Think

A sample game of tick-tack-toe is shown to the right. One player uses an **x** and the other player uses an **o**. Taking alternate turns, each tries to win the game by getting a complete row, column, or diagonal of his symbol.

x plays first.



In each diagram below, assume that you are using the **x**. If it is your turn to play, show how you can be sure of winning the game.



8-5 IMPROPER FRACTIONS AND MIXED NUMERALS

An **improper fraction** is a fraction whose numerator is greater than or equal to its denominator. The fractions $\frac{7}{4}$, $\frac{9}{5}$, $\frac{8}{8}$, and $\frac{15}{2}$ are examples of improper fractions. Whether we write an improper fraction or a mixed numeral usually depends on the kind of problem involved.

The exercises will help you review mixed numerals and improper fractions.

Exercises

1. Give the missing numerators of the fractions.

[a] $3 = \frac{\square}{1} = \frac{\square}{2} = \frac{\square}{3}$

[b] $7 = \frac{\square}{3} = \frac{\square}{5} = \frac{\square}{7}$

[c] $10 = \frac{\square}{4} = \frac{\square}{8} = \frac{\square}{10}$

[d] $13 = \frac{\square}{1} = \frac{\square}{2} = \frac{\square}{3}$

[e] $9 = \frac{\square}{7} = \frac{\square}{8} = \frac{\square}{9}$

[f] $0 = \frac{\square}{5} = \frac{\square}{11} = \frac{\square}{20}$

2. Give the numerator a, the whole number for b, and the mixed numeral for c.

[a] $\frac{5}{3} = \frac{a}{3} + \frac{2}{3} \longrightarrow \frac{5}{3} = b + \frac{2}{3} \longrightarrow \frac{5}{3} = c$

[b] $\frac{15}{4} = \frac{a}{4} + \frac{3}{4} \longrightarrow \frac{15}{4} = b + \frac{3}{4} \longrightarrow \frac{15}{4} = c$

[c] $\frac{23}{8} = \frac{a}{8} + \frac{7}{8} \longrightarrow \frac{23}{8} = b + \frac{7}{8} \longrightarrow \frac{23}{8} = c$

[d] $\frac{47}{6} = \frac{a}{6} + \frac{5}{6} \longrightarrow \frac{47}{6} = b + \frac{5}{6} \longrightarrow \frac{47}{6} = c$

3. Give the numerators r and s.

[a] $6\frac{1}{5} = \frac{r}{5} + \frac{1}{5} \longrightarrow 6\frac{1}{5} = \frac{s}{5}$

[b] $5\frac{1}{4} = \frac{r}{4} + \frac{1}{4} \longrightarrow 5\frac{1}{4} = \frac{s}{4}$

[c] $3\frac{2}{3} = \frac{r}{3} + \frac{2}{3} \longrightarrow 3\frac{2}{3} = \frac{s}{3}$

[d] $7\frac{4}{5} = \frac{r}{5} + \frac{4}{5} \longrightarrow 7\frac{4}{5} = \frac{s}{5}$

4. Give the numerator a and then give the improper fraction for b.

[a] $3\frac{1}{2} = \frac{a}{2} + \frac{1}{2} = b$

[b] $6\frac{1}{4} = \frac{a}{4} + \frac{1}{4} = b$

[c] $3\frac{2}{5} = \frac{a}{5} + \frac{2}{5} = b$

[d] $10\frac{1}{2} = \frac{a}{2} + \frac{1}{2} = b$

[e] $4\frac{3}{8} = \frac{a}{8} + \frac{3}{8} = b$

[f] $6\frac{2}{3} = \frac{a}{3} + \frac{2}{3} = b$

[g] $7\frac{5}{6} = \frac{a}{6} + \frac{5}{6} = b$

[h] $12\frac{3}{4} = \frac{a}{4} + \frac{3}{4} = b$

[i] $8\frac{4}{9} = \frac{a}{9} + \frac{4}{9} = b$

5. Give the missing numerators.

[a] $\frac{5}{3} = 1 + \frac{\square}{3}$

[b] $\frac{16}{9} = 1 + \frac{\square}{9}$

[c] $\frac{12}{5} = 2 + \frac{\square}{5}$

[d] $\frac{13}{3} = 4 + \frac{\square}{3}$

[e] $\frac{35}{8} = 4 + \frac{\square}{8}$

[f] $\frac{65}{6} = 10 + \frac{\square}{6}$

6. Give the numerators a and b.

[a] $4\frac{1}{8} = 3 + \left(\frac{a}{8} + \frac{1}{8}\right) = 3\frac{b}{8}$

[b] $6\frac{1}{4} = 5 + \left(\frac{a}{4} + \frac{1}{4}\right) = 5\frac{b}{4}$

[c] $3\frac{2}{3} = 2 + \left(\frac{a}{3} + \frac{2}{3}\right) = 2\frac{b}{3}$

[d] $7\frac{4}{5} = 6 + \left(\frac{a}{5} + \frac{4}{5}\right) = 6\frac{b}{5}$

7. Solve the equations. Reduce your answers to lowest terms.

[a] $4 + n = 3\frac{5}{4}$

[b] $10 + n = 9\frac{11}{8}$

[c] $12\frac{12}{9} = 13 + n$

[d] $23 + n = 22\frac{18}{16}$

[e] $30\frac{7}{4} = 31 + n$

[f] $72\frac{19}{16} = 73 + n$

8. Write an improper fraction for each mixed numeral.

[a] $7\frac{1}{8}$ [b] $2\frac{5}{6}$ [c] $8\frac{2}{3}$ [d] $33\frac{1}{3}$ [e] $6\frac{3}{4}$ [f] $10\frac{9}{10}$ [g] $37\frac{1}{2}$ [h] $18\frac{3}{7}$ [i] $16\frac{2}{3}$ [j] $11\frac{3}{7}$ [k] $30\frac{1}{4}$ [l] $62\frac{1}{2}$

9. Write a mixed numeral for each improper fraction.

[a] $\frac{7}{3}$ [b] $\frac{13}{5}$ [c] $\frac{27}{10}$ [d] $\frac{25}{4}$ [e] $\frac{47}{12}$ [f] $\frac{27}{2}$ [g] $\frac{39}{8}$ [h] $\frac{66}{7}$ [i] $\frac{74}{9}$ [j] $\frac{31}{16}$ [k] $\frac{111}{20}$ [l] $\frac{51}{13}$ [m] $\frac{49}{10}$ [n] $\frac{223}{100}$ [o] $\frac{5287}{1000}$ [p] $\frac{610}{80}$ [q] $\frac{75}{24}$ [r] $\frac{90}{60}$

8-6 BASIC PRINCIPLES FOR ADDITION OF RATIONAL NUMBERS

Commutative principle	Associative principle	Zero principle
The sum of two rational numbers is the same even if the order of the addends is changed.	The sum of three rational numbers is the same even if the grouping of the addends is changed.	The sum of any rational number and zero is that rational number.
$\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$	$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$	$\frac{a}{b}+0=\frac{a}{b}$

Since addition is both commutative and associative, we may arrange the addends in any way we choose.

$$\frac{1}{3}+\frac{3}{4}+\frac{5}{6}=\frac{5}{6}+\frac{1}{3}+\frac{3}{4}$$

We say that zero is the **identity element** for addition.

Exercises

1. Use the commutative, associative, and zero principles to find the sums.

[a] $\frac{1}{6}+\left(\frac{5}{6}+\frac{1}{4}\right)$ [c] $\frac{3}{10}+\left(\frac{7}{10}+\frac{1}{3}\right)$ [e] $\frac{5}{6}+\left(\frac{9}{10}+\frac{1}{6}\right)$ [g] $\left(\frac{6}{13}+\frac{0}{11}\right)+\frac{2}{13}$

[b] $\left(\frac{3}{8}+\frac{3}{4}\right)+\frac{1}{4}$ [d] $\left(\frac{1}{2}+0\right)+\frac{1}{2}$ [f] $\left(\frac{7}{11}+\frac{5}{9}\right)+\frac{4}{11}$ [h] $\frac{3}{7}+\left(\frac{2}{7}+\frac{0}{3}\right)$

2. Solve the equations.

[a] $\frac{5}{12}+\left(\frac{3}{8}+\frac{7}{12}\right)=1+n$

[b] $\left(\frac{3}{11}+\frac{7}{9}\right)+\frac{2}{9}=n+\frac{3}{11}$

[c] $\left(\frac{3}{4}+\frac{1}{2}\right)+\frac{1}{4}=n+1$

[d] $\left(3+\frac{1}{2}\right)+\left(1+\frac{1}{2}\right)=(3+1)+\left(\frac{1}{2}+n\right)$

[e] $\left(2+\frac{3}{4}\right)+\left(5+\frac{1}{8}\right)=\left(n+\frac{3}{4}\right)+\frac{1}{8}$

[f] $\left(6+\frac{1}{3}\right)+\left(4+\frac{1}{4}\right)=\left(10+\frac{1}{4}\right)+n$

[g] $\left(2+\frac{9}{16}\right)+\left(3+\frac{1}{16}\right)=5+n$

[h] $\left(6+\frac{3}{10}\right)+(1+n)=7+\frac{4}{10}$

[i] $3\frac{1}{8}+6\frac{3}{8}=9+n$

[j] $2\frac{4}{9}+3\frac{1}{9}=n+\frac{5}{9}$

[k] $7\frac{1}{6}+n=8\frac{5}{6}$

[l] $(n+2)+\left(2\frac{2}{5}\right)=5$

In Exercises 3 through 5, find the sums and differences. Use mixed numerals and lowest-terms fractions for your answers.

3.

Example:

$$\begin{aligned} \tfrac{5}{8} &= \tfrac{15}{24} \\ +\tfrac{5}{6} &= \tfrac{20}{24} \\ \hline \tfrac{35}{24} &= 1\tfrac{11}{24} \end{aligned}$$

[a] $\begin{array}{r} \frac{3}{4} \\ +\frac{2}{3} \\ \hline \end{array}$ [b] $\begin{array}{r} \frac{9}{10} \\ +\frac{3}{5} \\ \hline \end{array}$ [c] $\begin{array}{r} \frac{5}{12} \\ +\frac{5}{8} \\ \hline \end{array}$ [d] $\begin{array}{r} \frac{1}{8} \\ +\frac{8}{9} \\ \hline \end{array}$ [e] $\begin{array}{r} \frac{7}{15} \\ +\frac{3}{5} \\ \hline \end{array}$

[f] $\begin{array}{r} \frac{8}{9} \\ +\frac{5}{12} \\ \hline \end{array}$ [g] $\begin{array}{r} \frac{5}{8} \\ +\frac{3}{4} \\ \hline \end{array}$ [h] $\begin{array}{r} \frac{7}{16} \\ +\frac{3}{4} \\ \hline \end{array}$ [i] $\begin{array}{r} \frac{5}{8} \\ +\frac{2}{3} \\ \hline \end{array}$ [j] $\begin{array}{r} \frac{1}{2} \\ +\frac{4}{7} \\ \hline \end{array}$

4.

Example:

$$\begin{aligned} 7\tfrac{1}{4} &= 7\tfrac{2}{8} \\ +3\tfrac{7}{8} &= 3\tfrac{7}{8} \\ \hline 10\tfrac{9}{8} &= 11\tfrac{1}{8} \end{aligned}$$

[a] $\begin{array}{r} 3\frac{9}{10} \\ +4\frac{4}{5} \\ \hline \end{array}$ [b] $\begin{array}{r} 11\frac{7}{10} \\ +16\frac{3}{4} \\ \hline \end{array}$ [c] $\begin{array}{r} 23\frac{9}{10} \\ +32\frac{1}{5} \\ \hline \end{array}$ [d] $\begin{array}{r} 83\frac{1}{3} \\ +16\frac{2}{3} \\ \hline \end{array}$ [e] $\begin{array}{r} 52\frac{1}{2} \\ +29\frac{7}{10} \\ \hline \end{array}$

[f] $\begin{array}{r} 41\frac{1}{8} \\ +33\frac{5}{6} \\ \hline \end{array}$ [g] $\begin{array}{r} 19\frac{5}{9} \\ +20\frac{5}{6} \\ \hline \end{array}$ [h] $\begin{array}{r} 77\frac{1}{6} \\ +34\frac{2}{3} \\ \hline \end{array}$ [i] $\begin{array}{r} 74\frac{3}{8} \\ +62\frac{5}{6} \\ \hline \end{array}$ [j] $\begin{array}{r} 7\frac{13}{14} \\ +8\frac{1}{2} \\ \hline \end{array}$

5.

Example:

$$\begin{aligned} 27\tfrac{1}{3} &= 27\tfrac{2}{6} = 26\tfrac{8}{6} \\ -18\tfrac{5}{6} &= -18\tfrac{5}{6} = -18\tfrac{5}{6} \\ \hline 8\tfrac{3}{6} &= 8\tfrac{1}{2} \end{aligned}$$

[a] $\begin{array}{r} 7\frac{7}{8} \\ -3\frac{3}{4} \\ \hline \end{array}$ [b] $\begin{array}{r} 16\frac{1}{4} \\ -12\frac{3}{8} \\ \hline \end{array}$ [c] $\begin{array}{r} 35\frac{1}{6} \\ -28\frac{1}{4} \\ \hline \end{array}$ [d] $\begin{array}{r} 8\frac{1}{5} \\ -7\frac{9}{10} \\ \hline \end{array}$

[e] $\begin{array}{r} 11\frac{1}{10} \\ -5\frac{7}{10} \\ \hline \end{array}$ [f] $\begin{array}{r} 14\frac{1}{2} \\ -8\frac{2}{3} \\ \hline \end{array}$ [g] $\begin{array}{r} 19\frac{3}{8} \\ -13\frac{3}{4} \\ \hline \end{array}$ [h] $\begin{array}{r} 66\frac{2}{3} \\ -33\frac{3}{4} \\ \hline \end{array}$

Think

Find the sums for Exercises 1 through 4. Use these results to find the sum for Exercise 5.

1. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}$

2. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}$

3. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\dfrac{1}{4\cdot 5}$

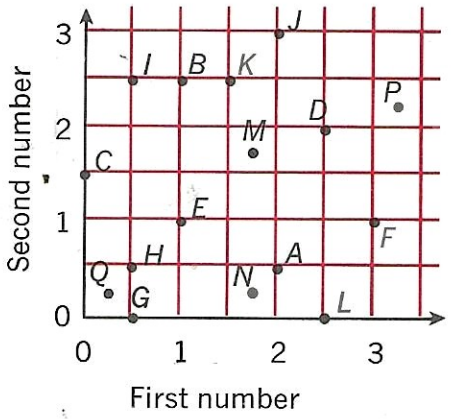
4. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\dfrac{1}{4\cdot 5}+\dfrac{1}{5\cdot 6}$

5. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{99\cdot 100}$

8-7 GRAPHING PAIRS OF RATIONAL NUMBERS

With each point shown on the grid, we can match a pair of rational numbers.

Study the graph and the table to see how the points and number pairs are matched.



Number pair	Point for the pair
$(2, \frac{1}{2})$	A
$(1, 2\frac{1}{2})$	B
$(0, 1\frac{1}{2})$	C
$(2\frac{1}{2}, 2)$	D
$(1, 1)$	E
$(3, 1)$	F
$(\frac{1}{2}, 0)$	G

Discussion Exercises

1. From the graph above, give the number pair for points H, I, J, K, and L.

2. Estimate the number pair for points M, N, P, and Q.

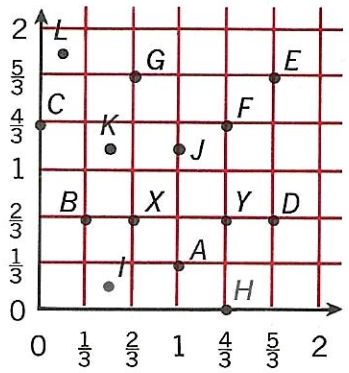
Exercises

1. [a] Give the number pair for points A, B, C, D, E, F, G, and H.

   [b] Estimate the number pair for each point I, J, K, and L.

   [c] What do the number pairs for points B, D, X, and Y have in common?

   [d] What can be said about all the number pairs for the other points on the line through B, D, X, and Y?



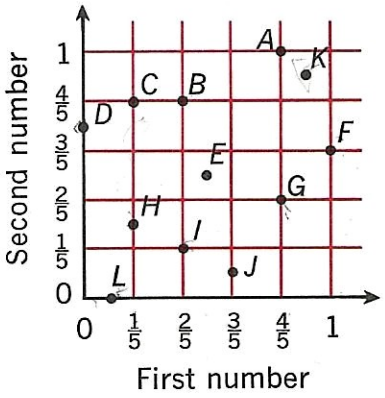
2. For each ▒, write the point for the number pair.

[a] $(\frac{4}{5}, \frac{2}{5})$, ▒ [e] $(\frac{2}{5}, \frac{1}{5})$, ▒ [i] $(\frac{1}{5}, \frac{3}{10})$, ▒

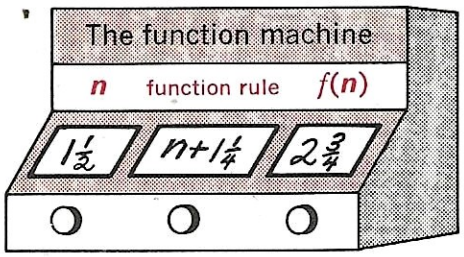
[b] $(\frac{1}{5}, \frac{4}{5})$, ▒ [f] $(\frac{2}{5}, \frac{4}{5})$, ▒ [j] $(\frac{1}{2}, \frac{1}{2})$, ▒

[c] $(\frac{4}{5}, 1)$, ▒ [g] $(\frac{3}{5}, \frac{1}{10})$, ▒ [k] $(\frac{1}{10}, 0)$, ▒

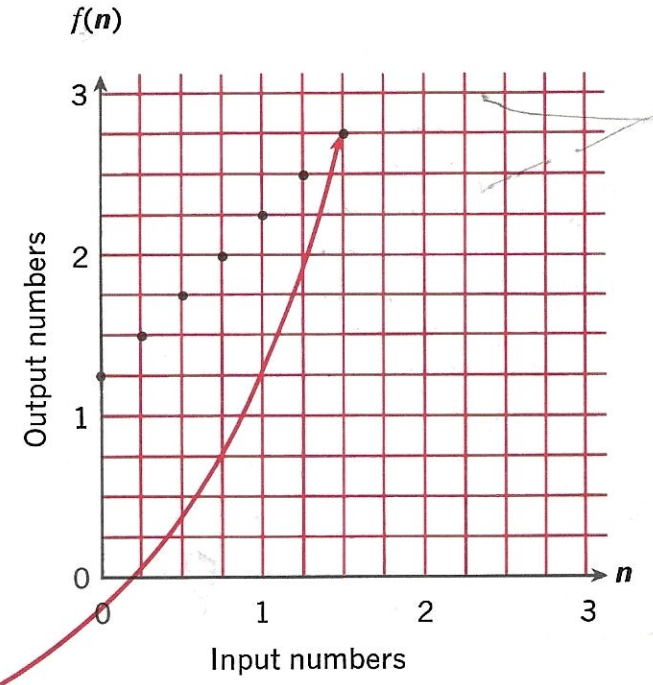
[d] $(1, \frac{3}{5})$, ▒ [h] $(0, \frac{7}{10})$, ▒ [l] $(\frac{9}{10}, \frac{9}{10})$, ▒



3. Study the table and the graph of each number pair.



function rule $n + 1\frac{1}{4}$			Number pairs
n	$f(n)$		
0	$1\frac{1}{4}$	→	$(0, 1\frac{1}{4})$
$\frac{1}{4}$	$1\frac{1}{2}$	→	$(\frac{1}{4}, 1\frac{1}{2})$
$\frac{1}{2}$	$1\frac{3}{4}$	→	$(\frac{1}{2}, 1\frac{3}{4})$
$\frac{3}{4}$	2	→	$(\frac{3}{4}, 2)$
1	$2\frac{1}{4}$	→	$(1, 2\frac{1}{4})$
$1\frac{1}{4}$	$2\frac{1}{2}$	→	$(1\frac{1}{4}, 2\frac{1}{2})$
$1\frac{1}{2}$	$2\frac{3}{4}$	→	$(1\frac{1}{2}, 2\frac{3}{4})$



[a] Copy the function table on the right and find the output numbers.

[b] Do the graphs of the number pairs follow the pattern shown on the grid?

function rule $n + 1\frac{1}{4}$	
n	$f(n)$
$1\frac{3}{4}$	
2	
$2\frac{1}{4}$	

4. Draw and label two number lines like the ones in Exercise 3. Graph the number pairs in the following table.

function rule $n + \frac{1}{2}$	
n	$f(n)$
0	$\frac{1}{2}$
$\frac{1}{2}$	1
1	$1\frac{1}{2}$
$1\frac{1}{2}$	2
2	$2\frac{1}{2}$
$2\frac{1}{2}$	3

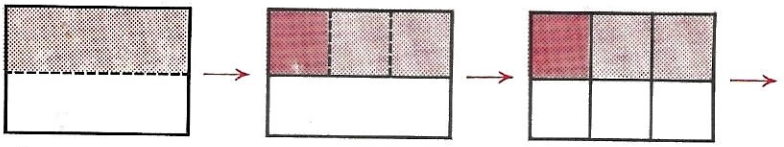
5. Copy and complete the following table. Then graph the number pairs on the grid you drew for Exercise 4.

function rule $n + n$	
n	$f(n)$
0	
$\frac{1}{4}$	
$\frac{1}{2}$	
$\frac{3}{4}$	
1	
$1\frac{1}{4}$	

8-8 REGIONS, NUMBER LINES, AND MULTIPLICATION

The examples will help you review the basic ideas related to multiplication of rational numbers.

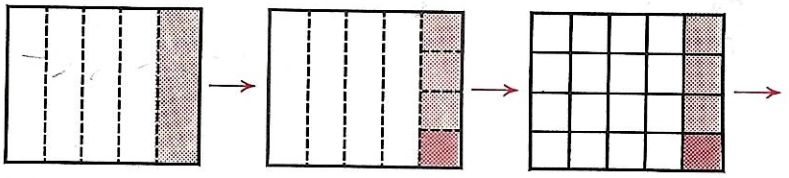
A



$\frac{1}{2}$ of a region → $\frac{1}{3}$ of $\frac{1}{2}$ → $\frac{1}{6}$ of a region

We write: $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

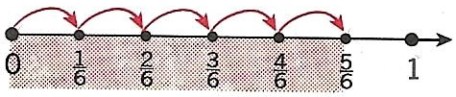
B



$\frac{1}{5}$ of a region → $\frac{1}{4}$ of $\frac{1}{5}$ → $\frac{1}{20}$ of a region

We write: $\frac{1}{4} \cdot \frac{1}{5} = \frac{1}{20}$

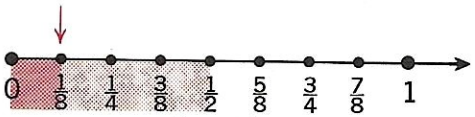
C



5 jumps of $\frac{1}{6}$

We write: $5 \cdot \frac{1}{6} = \frac{5}{6}$.

D



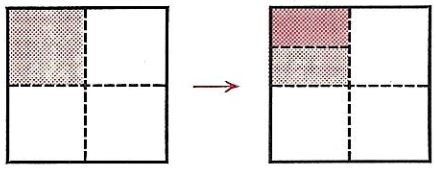
$\frac{1}{4}$ of $\frac{1}{2}$

We write: $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$.

Exercises

1. Write the multiplication equation suggested by each exercise.

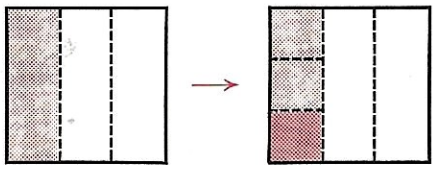
Example:



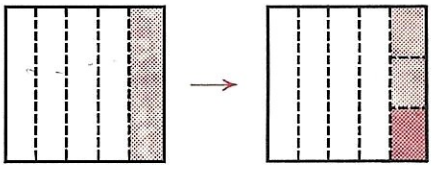
$\frac{1}{4}$ of a region $\frac{1}{2}$ of $\frac{1}{4}$

$\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

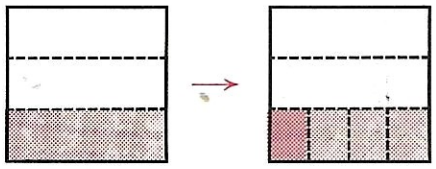
[a]



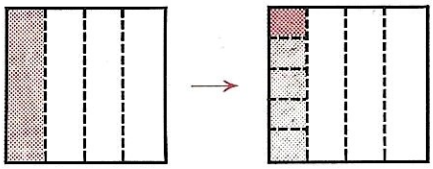
[b]



[c]



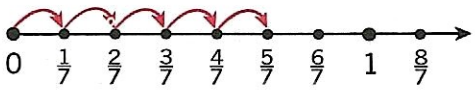
[d]



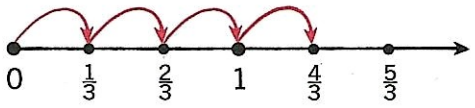
2. Draw two squares. Divide and shade them as in Exercise 1 to illustrate the product $\frac{1}{4} \cdot \frac{1}{4}$.

3. Give the multiplication fact suggested by the number line.

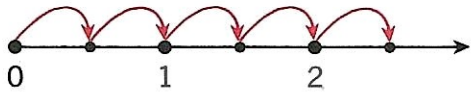
[a]



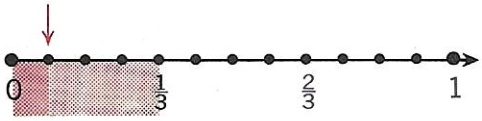
[b]



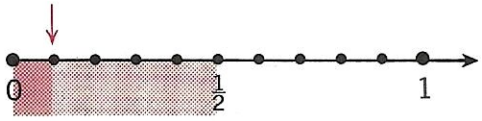
[c]



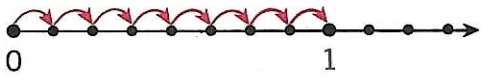
[d]



[e]



[f]



4. Solve the equations.

[a] $\frac{1}{2} \cdot \frac{1}{3} = n$
[b] $\frac{1}{4} \cdot \frac{1}{5} = n$
[c] $\frac{1}{3} \cdot \frac{1}{3} = n$
[d] $\frac{1}{5} \cdot \frac{1}{3} = n$
[e] $\frac{1}{5} \cdot \frac{1}{5} = n$
[f] $\frac{1}{2} \cdot \frac{1}{4} = n$
[g] $\frac{1}{6} \cdot \frac{1}{7} = n$
[h] $\frac{1}{6} \cdot \frac{1}{6} = n$
[i] $\frac{1}{3} \cdot \frac{1}{6} = n$
[j] $\frac{1}{5} \cdot \frac{1}{4} = n$
[k] $\frac{1}{6} \cdot \frac{1}{5} = n$
[l] $\frac{1}{8} \cdot \frac{1}{5} = n$
[m] $\frac{1}{10} \cdot \frac{1}{10} = n$
[n] $\frac{1}{10} \cdot \frac{1}{100} = n$
[o] $\frac{1}{10} \cdot \frac{1}{1000} = n$
[p] $\frac{1}{100} \cdot \frac{1}{100} = n$

5. Find the products.

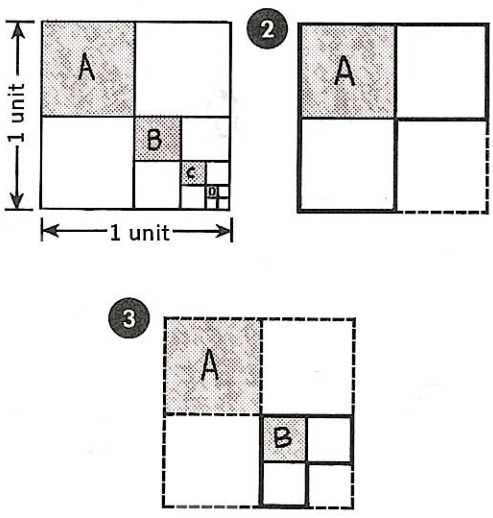
[a] $3 \cdot \frac{1}{4}$
[b] $5 \cdot \frac{1}{8}$
[c] $7 \cdot \frac{1}{9}$
[d] $7 \cdot \frac{1}{6}$
[e] $10 \cdot \frac{1}{10}$
[f] $4 \cdot \frac{1}{2}$
[g] $12 \cdot \frac{1}{5}$
[h] $9 \cdot \frac{1}{9}$
[i] $10 \cdot \frac{1}{100}$
[j] $7 \cdot \frac{1}{1000}$
[k] $13 \cdot \frac{1}{11}$
[l] $14 \cdot \frac{1}{5}$
[m] $0 \cdot \frac{1}{3}$
[n] $1 \cdot \frac{1}{8}$
[o] $17 \cdot \frac{1}{17}$
★ [p] $a \cdot \frac{1}{a}$, $a \neq 0$

6. Find the number for a. Then find the number for b.

[a] $\frac{1}{2} \cdot \frac{1}{5} = a \longrightarrow \frac{1}{5} \cdot \frac{1}{2} = b$
[b] $\frac{1}{3} \cdot \frac{1}{7} = a \longrightarrow \frac{1}{7} \cdot \frac{1}{3} = b$
[c] $\frac{1}{8} \cdot \frac{1}{9} = a \longrightarrow \frac{1}{9} \cdot \frac{1}{8} = b$
[d] $\frac{1}{2} \cdot (\frac{1}{3} \cdot \frac{1}{4}) = a \longrightarrow (\frac{1}{2} \cdot \frac{1}{3}) \cdot \frac{1}{4} = b$
[e] $\frac{1}{5} \cdot (\frac{1}{2} \cdot \frac{1}{3}) = a \longrightarrow (\frac{1}{5} \cdot \frac{1}{2}) \cdot \frac{1}{3} = b$
[f] $\frac{1}{2} \cdot (\frac{1}{4} \cdot \frac{1}{5}) = a \longrightarrow (\frac{1}{2} \cdot \frac{1}{4}) \cdot \frac{1}{5} = b$

Think

1. Give rational numbers for the areas of the shaded regions *A*, *B*, *C*, and *D*.
2. Find the sum of the numbers for the regions *A*, *B*, *C*, and *D*.
3. Is the sum of Exercise 2 greater than, equal to, or less than $\frac{1}{3}$?
4. In Figure 2, what part of the L-shaped region is *A*?
5. In Figure 3, what part of the L-shaped region is region B?
6. If the dividing and shading process shown in Figure 1 were carried on without end, what part of the unit square would never be shaded?



8-9 BASIC PRINCIPLES FOR RATIONAL NUMBERS

For addition and multiplication of rational numbers, we accept the same principles that we have used for whole numbers. We will now use these principles for computing products of rational numbers.

Basic principles for rational numbers		
Addition		**Multiplication**
The **sum** of two rational numbers is the same even if the order of the **addends** is changed. $a + b = b + a$	**Commutative principle**	The **product** of two rational numbers is the same even if the order of the **factors** is changed. $a \cdot b = b \cdot a$
The **sum** of three rational numbers is the same even if the grouping of the **addends** is changed. $a + (b + c) = (a + b) + c$	**Associative principle**	The **product** of three rational numbers is the same even if the grouping of the **factors** is changed. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
The **sum** of any rational number and 0 is that rational number. We say that 0 is the **identity element** in addition. $a + 0 = a = 0 + a$	**Identity principles** 0 principle ← | → 1 principle	The **product** of any rational number and 1 is that rational number. We say that 1 is the **identity element** in multiplication. $a \cdot 1 = a = 1 \cdot a$

Since we can change the order and the grouping of factors, we can arrange the factors in any way that is convenient.

Oral Exercises

1. Give the products.

[a] $\frac{5}{7}\cdot\frac{9}{9}$ [b] $\frac{13}{25}\cdot\frac{47}{47}$ [c] $\frac{73}{42}\cdot\frac{0}{61}$ [d] $\left(\frac{7}{11}\cdot\frac{5}{9}\right)\cdot\frac{0}{23}$ [e] $\frac{31}{31}\cdot\frac{0}{25}$ [f] $\frac{67}{67}\cdot\frac{99}{99}$

2. Use the commutative principle to find n.

[a] Since $\frac{3}{5}\cdot\frac{2}{3}=\frac{2}{5}$, then $\frac{2}{3}\cdot\frac{3}{5}=n$.

[b] Since $\frac{1}{2}\cdot\frac{4}{7}=\frac{2}{7}$, then $\frac{4}{7}\cdot\frac{1}{2}=n$.

[c] Since $\frac{3}{4}\cdot\frac{6}{5}=\frac{9}{10}$, then $\frac{6}{5}\cdot\frac{3}{4}=n$.

[d] Since $\frac{2}{5}\cdot\frac{7}{10}=\frac{7}{25}$, then $\frac{7}{10}\cdot\frac{2}{5}=n$.

[e] Since $\frac{8}{3}\cdot\frac{5}{8}=\frac{5}{3}$, then $\frac{5}{8}\cdot\frac{8}{3}=n$.

3. Find answers in the easiest way possible.

[a] $\frac{3}{5}\cdot\left(\frac{3}{4}+\frac{1}{4}\right)$ [b] $\frac{6}{7}\cdot\left(\frac{5}{8}+\frac{3}{8}\right)$ [c] $\left(\frac{15}{19}+\frac{4}{19}\right)\cdot\frac{17}{23}$ [d] $\left(\frac{23}{39}+\frac{16}{39}\right)\cdot\frac{37}{45}$

Exercises

1. Find the number for a. Then find the number for b.

[a] $3\cdot\frac{1}{4}=a\longrightarrow\frac{1}{4}\cdot 3=b$

[b] $5\cdot\frac{1}{7}=a\longrightarrow\frac{1}{7}\cdot 5=b$

[c] $8\cdot\frac{1}{3}=a\longrightarrow\frac{1}{3}\cdot 8=b$

[d] $10\cdot\frac{1}{10}=a\longrightarrow\frac{1}{10}\cdot 10=b$

[e] $11\cdot\frac{1}{100}=a\longrightarrow\frac{1}{100}\cdot 11=b$

[f] $7\cdot\frac{1}{4}=a\longrightarrow\frac{1}{4}\cdot 7=b$

2. Solve the equations.

[a] $\frac{1}{4}\cdot 1=n$ [b] $1\cdot\frac{1}{8}=n$ [c] $n\cdot 1=\frac{5}{6}$ [d] $\frac{3}{5}\cdot\frac{3}{3}=n$ [e] $\frac{5}{2}\cdot\frac{4}{4}=n$ [f] $\frac{9}{10}\cdot\frac{10}{10}=n$ [g] $\frac{1}{9}\cdot n=\frac{1}{9}$ [h] $\frac{0}{3}\cdot\frac{1}{4}=n$ [i] $\frac{5}{12}\cdot n=0$ [j] $n\cdot\frac{5}{5}=\frac{7}{8}$

3. Solve the equations.

[a] $3\cdot\left(\frac{1}{2}\cdot\frac{1}{4}\right)=n$ [b] $\left(3\cdot\frac{1}{2}\right)\cdot\frac{1}{4}=n$ [c] $\left(3\cdot\frac{1}{4}\right)\cdot\frac{1}{2}=n$ [d] $5\cdot\left(\frac{1}{3}\cdot\frac{1}{4}\right)=n$ [e] $\left(5\cdot\frac{1}{4}\right)\cdot\frac{1}{3}=n$ [f] $\left(5\cdot\frac{1}{3}\right)\cdot\frac{1}{4}=n$ [g] $7\cdot\left(\frac{1}{2}\cdot\frac{1}{5}\right)=n$ [h] $\left(7\cdot\frac{1}{5}\right)\cdot\frac{1}{2}=n$

4. Solve the equations.

[a] $\left(6\cdot\frac{1}{4}\right)\cdot\frac{1}{3}=6\cdot\left(\frac{1}{4}\cdot n\right)$ [b] $\left(7\cdot\frac{1}{5}\right)\cdot n=7\cdot\left(\frac{1}{5}\cdot\frac{1}{8}\right)$ [c] $\left(n\cdot\frac{1}{2}\right)\cdot\frac{1}{5}=7\cdot\left(\frac{1}{2}\cdot\frac{1}{5}\right)$

5. Find each product by first finding the product of the unit fractions.

[a] $3\cdot\frac{1}{2}\cdot\frac{1}{3}$ [b] $7\cdot\frac{1}{5}\cdot\frac{1}{6}$ [c] $\frac{1}{4}\cdot 5\cdot\frac{1}{2}$ [d] $\frac{1}{10}\cdot\frac{1}{10}\cdot 100$ [e] $7\cdot\frac{1}{3}\cdot\frac{1}{5}$ [f] $3\cdot\frac{1}{10}\cdot\frac{1}{10}$ [g] $23\cdot\frac{1}{10}\cdot\frac{1}{100}$ [h] $\frac{1}{4}\cdot 5\cdot\frac{1}{4}$ [i] $\frac{1}{3}\cdot 4\cdot\frac{1}{7}$ [j] $\frac{1}{8}\cdot 24\cdot\frac{1}{3}$

6. Solve the equations.

[a] $4 \cdot \frac{1}{8} \cdot 2 \cdot \frac{1}{3} = n \cdot (\frac{1}{8} \cdot \frac{1}{3})$

[b] $3 \cdot \frac{1}{5} \cdot 2 \cdot \frac{1}{7} = n \cdot (\frac{1}{5} \cdot \frac{1}{7})$

[c] $10 \cdot \frac{1}{3} \cdot 10 \cdot \frac{1}{3} = n \cdot \frac{1}{9}$

[d] $6 \cdot \frac{1}{5} \cdot 3 \cdot \frac{1}{4} = (6 \cdot 3) \cdot n$

[e] $5 \cdot \frac{1}{2} \cdot 7 \cdot \frac{1}{3} = 35 \cdot n$

[f] $11 \cdot \frac{1}{4} \cdot 5 \cdot \frac{1}{2} = n \cdot \frac{1}{8}$

Think

If the number of eggs in a crate are counted by twos, threes, fives, or sevens, there is one egg left over each time. What is the smallest number of eggs that could be in the crate?

8-10 FINDING PRODUCTS

By using the basic principles, we have a way to find products such as $\frac{3}{7} \cdot \frac{4}{5}$. First we combine associative and commutative principles to arrange the factors in any convenient way. Then we can find the product of these factors.

$$\frac{3}{7} \cdot \frac{4}{5} = (3 \cdot \tfrac{1}{7}) \cdot (4 \cdot \tfrac{1}{5}) = 12 \cdot \tfrac{1}{35} = \tfrac{12}{35}$$

Exercises

1. Give the whole number for ***a*** and the unit fraction for ***b***. Then give the missing product for ***c***.

Example: $\frac{5}{9} \cdot \frac{2}{3} = (5 \cdot \frac{1}{9}) \cdot (2 \cdot \frac{1}{3}) = 10 \cdot \frac{1}{27} = \frac{10}{27}$

[a] $\frac{3}{4} \cdot \frac{5}{2} = (3 \cdot \frac{1}{4}) \cdot (5 \cdot \frac{1}{2}) = a \cdot b = c$

[b] $\frac{3}{8} \cdot \frac{3}{4} = (3 \cdot \frac{1}{8}) \cdot (3 \cdot \frac{1}{4}) = a \cdot b = c$

[c] $\frac{6}{11} \cdot \frac{1}{2} = (6 \cdot \frac{1}{11}) \cdot (1 \cdot \frac{1}{2}) = a \cdot b = c$

[d] $\frac{2}{5} \cdot \frac{3}{4} = (2 \cdot \frac{1}{5}) \cdot (3 \cdot \frac{1}{4}) = a \cdot b = c$

[e] $\frac{3}{4} \cdot \frac{4}{3} = (3 \cdot \frac{1}{4}) \cdot (4 \cdot \frac{1}{3}) = a \cdot b = c$

[f] $\frac{7}{2} \cdot \frac{3}{2} = (7 \cdot \frac{1}{2}) \cdot (3 \cdot \frac{1}{2}) = a \cdot b = c$

[g] $\frac{7}{10} \cdot \frac{3}{10} = (7 \cdot \frac{1}{10}) \cdot (3 \cdot \frac{1}{10}) = a \cdot b = c$

[h] $\frac{10}{3} \cdot \frac{9}{2} = (10 \cdot \frac{1}{3}) \cdot (9 \cdot \frac{1}{2}) = a \cdot b = c$

2. Find the products.

[a] $\frac{2}{3}\cdot\frac{3}{4}=(2\cdot3)\cdot(\frac{1}{3}\cdot\frac{1}{4})=n$

[b] $\frac{4}{7}\cdot\frac{5}{8}=(4\cdot5)\cdot(\frac{1}{7}\cdot\frac{1}{8})=n$

[c] $\frac{12}{17}\cdot\frac{7}{11}=(12\cdot7)\cdot(\frac{1}{17}\cdot\frac{1}{11})=n$

[d] $\frac{9}{10}\cdot\frac{13}{10}=(9\cdot13)\cdot(\frac{1}{10}\cdot\frac{1}{10})=n$

[e] $\frac{7}{10}\cdot\frac{13}{100}=(7\cdot13)\cdot(\frac{1}{10}\cdot\frac{1}{100})=n$

[f] $\frac{47}{100}\cdot\frac{3}{10}=(47\cdot3)\cdot(\frac{1}{100}\cdot\frac{1}{10})=n$

3. Write each of the following as the product of one whole number and a unit fraction.

Example: $\frac{3}{5}\cdot\frac{2}{4}=6\cdot\frac{1}{20}$

[a] $\frac{2}{3}\cdot\frac{1}{5}$ [b] $\frac{3}{7}\cdot\frac{2}{3}$ [c] $\frac{4}{7}\cdot\frac{5}{6}$ [d] $\frac{7}{3}\cdot\frac{9}{8}$ [e] $\frac{7}{10}\cdot\frac{6}{10}$ [f] $\frac{11}{100}\cdot\frac{9}{10}$

4. Find the products.

[a] $\frac{3}{4}\cdot\frac{3}{5}$ [b] $\frac{2}{3}\cdot\frac{4}{5}$ [c] $\frac{3}{7}\cdot\frac{5}{6}$ [d] $\frac{6}{5}\cdot\frac{4}{3}$

[e] $\frac{7}{2}\cdot\frac{5}{3}$ [f] $\frac{7}{4}\cdot\frac{5}{2}$ [g] $\frac{5}{6}\cdot\frac{3}{8}$ [h] $\frac{3}{11}\cdot\frac{5}{13}$

[i] $\frac{53}{17}\cdot\frac{21}{34}$ [j] $\frac{3}{7}\cdot\frac{4}{19}$ [k] $\frac{4}{19}\cdot\frac{3}{7}$ [l] $\frac{13}{19}\cdot\frac{7}{7}$

[m] $\frac{43}{16}\cdot\frac{7}{16}$ [n] $\frac{7}{8}\cdot\frac{5}{6}$ [o] $\frac{16}{24}\cdot\frac{27}{36}$ [p] $\frac{10}{15}\cdot\frac{50}{100}$

[q] $\frac{17}{10}\cdot\frac{7}{10}$ [r] $\frac{23}{100}\cdot\frac{47}{100}$ [s] $\frac{5}{21}\cdot\frac{7}{25}$ ★ [t] $\frac{a}{b}\cdot\frac{c}{d}$

8-11 RECIPROCALS

Each example below shows a pair of rational numbers whose product is 1.

$$\frac{2}{3}\cdot\frac{3}{2}=\frac{6}{6}=1 \qquad \frac{3}{8}\cdot\frac{8}{3}=\frac{24}{24}=1$$

If the product of two rational numbers is 1, we say that each number is the reciprocal of the other. For example, $\frac{2}{3}$ is the reciprocal of $\frac{3}{2}$, and $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$.

Exercises

1. Find the products.

[a] $\frac{2}{1}\cdot\frac{1}{2}$ [b] $\frac{7}{4}\cdot\frac{4}{7}$ [c] $\frac{3}{5}\cdot\frac{5}{3}$ [d] $\frac{6}{7}\cdot\frac{7}{6}$ [e] $2\frac{1}{2}\cdot\frac{2}{5}$ [f] $\frac{7}{11}\cdot\frac{11}{7}$ [g] $\frac{13}{15}\cdot1\frac{2}{13}$ [h] $\frac{17}{100}\cdot\frac{100}{17}$ [i] $\frac{9}{1}\cdot\frac{1}{9}$ [j] $3\frac{1}{3}\cdot\frac{3}{10}$

2. Solve the equations by finding the reciprocal of the given factor.

[a] $n \cdot \frac{3}{8} = 1$ [c] $n \cdot \frac{9}{4} = 1$ [e] $3\frac{1}{2} \cdot n = 1$ [g] $\frac{1}{10} \cdot n = 1$

[b] $\frac{5}{6} \cdot n = 1$ [d] $n \cdot \frac{8}{7} = 1$ [f] $7 \cdot n = 1$ [h] $100 \cdot n = 1$

3. Give the reciprocals of the following numbers.

[a] $\frac{5}{8}$ [b] $\frac{8}{5}$ [c] $\frac{1}{9}$ [d] 8 [e] $\frac{4}{11}$ [f] $1\frac{7}{8}$ [g] $8\frac{1}{7}$ [h] $\frac{99}{100}$ [i] 10 [j] $\frac{3}{3}$

4. Find the number for ***a***. Then find the number for ***b***.

[a] $6 \cdot \frac{1}{6} = a \longrightarrow 12 \cdot \frac{1}{6} = b$

[b] $9 \cdot \frac{1}{9} = a \longrightarrow 27 \cdot \frac{1}{9} = b$

[c] $4 \cdot \frac{1}{4} = a \longrightarrow 16 \cdot \frac{1}{4} = b$

[d] $10 \cdot \frac{1}{10} = a \longrightarrow 100 \cdot \frac{1}{10} = b$

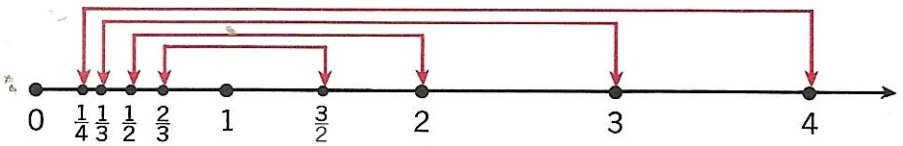
[e] $4 \cdot \frac{3}{4} = a \longrightarrow 8 \cdot \frac{3}{4} = b$

[f] $6 \cdot \frac{5}{6} = a \longrightarrow 24 \cdot \frac{5}{6} = b$

[g] $2 \cdot \frac{3}{2} = a \longrightarrow 10 \cdot \frac{3}{2} = b$

[h] $7 \cdot \frac{4}{7} = a \longrightarrow 21 \cdot \frac{4}{7} = b$

5. The number line shows how reciprocals are related. Study the number line and then give the number for each ▒.



[a] If a number is greater than 4, then its reciprocal is smaller than ▒.

[b] If a number is between 2 and 3, then its reciprocal will be between ▒ and ▒.

[c] Numbers between $\frac{3}{2}$ and 2 have reciprocals between ▒ and ▒.

[d] Numbers which are very large have reciprocals which are very close to ▒.

[e] ▒ has no reciprocal.

[f] Numbers which are very close to 1 have reciprocals which are very close to ▒.

[g] The reciprocal of a certain rational number is larger than the number. The number must have been less than ▒.

[h] ▒ is the only number which is its own reciprocal.

Think

Two numbers are reciprocals of each other. One of these numbers is four times as large as the other. Determine both numbers.

8-12 SHORTCUTS IN MULTIPLICATION

The following exercises will help you understand a shortcut for multiplying rational numbers.

Exercises

1. Find the products. Reduce each fraction to lowest terms before multiplying. Give answers in lowest terms.

[a] $\frac{3}{6}\cdot\frac{5}{20}$ [c] $\frac{11}{33}\cdot\frac{15}{30}$ [e] $\frac{12}{18}\cdot\frac{6}{14}$ [g] $\frac{24}{28}\cdot\frac{25}{90}$ [i] $\frac{70}{45}\cdot\frac{24}{28}$

[b] $\frac{4}{12}\cdot\frac{10}{20}$ [d] $\frac{10}{80}\cdot\frac{20}{100}$ [f] $\frac{12}{36}\cdot\frac{45}{10}$ [h] $\frac{77}{49}\cdot\frac{15}{110}$ [j] $\frac{41}{205}\cdot\frac{31}{93}$

2. Find the missing numerator or denominator.

[a] $\frac{1}{2}\cdot\frac{4}{5}=\frac{1\cdot4}{2\cdot5}=\frac{4\cdot1}{2\cdot5}=\frac{\blacksquare}{2}\cdot\frac{1}{5}$

[b] $\frac{2}{3}\cdot\frac{4}{5}=\frac{2\cdot4}{3\cdot5}=\frac{4\cdot2}{3\cdot5}=\frac{4}{3}\cdot\frac{2}{\blacksquare}$

[c] $\frac{6}{7}\cdot\frac{3}{8}=\frac{6\cdot3}{7\cdot8}=\frac{3\cdot6}{7\cdot8}=\frac{3}{\blacksquare}\cdot\frac{6}{8}$

[d] $\frac{3}{4}\cdot\frac{20}{9}=\frac{3\cdot20}{4\cdot9}=\frac{20\cdot3}{4\cdot9}=\frac{20}{4}\cdot\frac{\blacksquare}{9}$

[e] $\frac{15}{8}\cdot\frac{4}{9}=\frac{15\cdot4}{8\cdot9}=\frac{4\cdot15}{8\cdot9}=\frac{4}{\blacksquare}\cdot\frac{15}{9}$

★[f] $\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}=\frac{c\cdot a}{b\cdot d}=\frac{c}{b}\cdot\frac{\blacksquare}{d}$

3. Solve the equations.

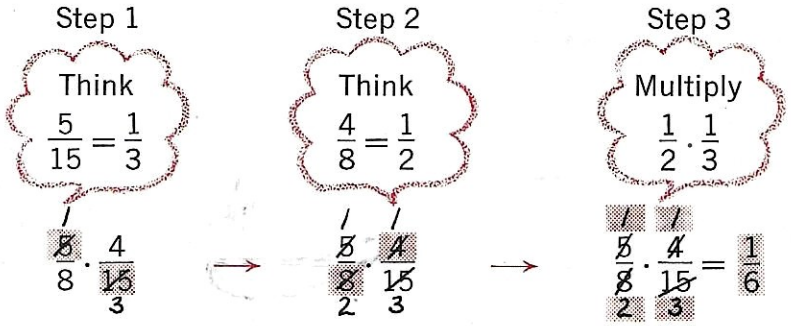
[a] $\frac{7}{8}\cdot\frac{5}{6}=\frac{5}{8}\cdot\frac{n}{6}$ [b] $\frac{6}{n}\cdot\frac{5}{12}=\frac{6}{12}\cdot\frac{5}{15}$ [c] $\frac{3}{4}\cdot\frac{5}{6}=\frac{n}{4}\cdot\frac{3}{6}$ [d] $\frac{2}{3}\cdot\frac{9}{8}=\frac{n}{3}\cdot\frac{2}{8}$

4. Find each product as in the example.

Example: $\frac{5}{9}\cdot\frac{3}{10}=\frac{3}{9}\cdot\frac{5}{10}=\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{6}$

[a] $\frac{3}{5}\cdot\frac{10}{9}$ [b] $\frac{5}{8}\cdot\frac{2}{15}$ [c] $\frac{3}{4}\cdot\frac{2}{9}$ [d] $\frac{7}{5}\cdot\frac{5}{8}$ [e] $\frac{3}{10}\cdot\frac{4}{3}$ [f] $\frac{7}{9}\cdot\frac{3}{14}$

Before trying Exercise 5, study the shortcut demonstrated in the following example.



5. Find the products, using the shortcut.

[a] $\frac{3}{8}\cdot\frac{4}{21}$ [c] $\frac{8}{21}\cdot\frac{14}{12}$ [e] $\frac{5}{7}\cdot\frac{21}{10}$ [g] $\frac{14}{63}\cdot\frac{15}{12}$ [i] $\frac{77}{32}\cdot\frac{12}{35}$

[b] $\frac{5}{4}\cdot\frac{2}{15}$ [d] $\frac{9}{20}\cdot\frac{25}{6}$ [f] $\frac{4}{9}\cdot\frac{15}{10}$ [h] $\frac{13}{38}\cdot\frac{19}{52}$ [j] $\frac{50}{42}\cdot\frac{14}{30}$

8-13 USING THE DISTRIBUTIVE PRINCIPLE

Multiplication and addition of rational numbers are related by the distributive principle

Examples: $4\cdot 32 = 4\cdot(30+2) = (4\cdot 30)+(4\cdot 2)$

$4\cdot 2\frac{1}{5} = 4\cdot(2+\frac{1}{5}) = (4\cdot 2)+(4\cdot\frac{1}{5})$

Study the following examples to see how the distributive principle can help you find products.

A

$5\cdot 3\frac{1}{6} = 5\cdot(3+\frac{1}{6}) = (5\cdot 3)+(5\cdot\frac{1}{6})$

Step 1	Step 2	Step 3
$3\frac{1}{6}$ $\times\ 5$ 15	$3\frac{1}{6}$ $\times\ 5$ $15\frac{5}{6}$	$3\frac{1}{6}$ $\times\ 5$ $15\frac{5}{6}$
$5\cdot 3 = 15$	$5\cdot\frac{1}{6} = \frac{5}{6}$	$15+\frac{5}{6} = 15\frac{5}{6}$

B

$\frac{1}{3}\cdot 6\frac{1}{4} = \frac{1}{3}\cdot(6+\frac{1}{4}) = \frac{1}{3}\cdot 6+\frac{1}{3}\cdot\frac{1}{4}$

Step 1	Step 2	Step 3
$6\frac{1}{4}$ $\times\ \frac{1}{3}$ 2	$6\frac{1}{4}$ $\times\ \frac{1}{3}$ $2\frac{1}{12}$	$6\frac{1}{4}$ $\times\ \frac{1}{3}$ $2\frac{1}{12}$
$\frac{1}{3}\cdot 6 = 2$	$\frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$	$2+\frac{1}{12} = 2\frac{1}{12}$

C

$2\frac{1}{2}\cdot 4\frac{1}{3} = (2+\frac{1}{2})\cdot 4\frac{1}{3} = 2\cdot 4\frac{1}{3}+\frac{1}{2}\cdot 4\frac{1}{3}$

Step 1	Step 2	Step 3
$4\frac{1}{3}$ $\times\ 2\frac{1}{2}$ $8\frac{2}{3}$	$4\frac{1}{3}$ $\times\ 2\frac{1}{2}$ $8\frac{2}{3}$ $2\frac{1}{6}$	$4\frac{1}{3}$ $\times\ 2\frac{1}{2}$ $8\frac{2}{3}$ $2\frac{1}{6}$ $10\frac{5}{6}$
$2\cdot 4\frac{1}{3} = 8\frac{2}{3}$	$\frac{1}{2}\cdot 4\frac{1}{3} = 2\frac{1}{6}$	$8\frac{2}{3}+2\frac{1}{6} = 10\frac{5}{6}$

Exercises

1. Use the distributive principle to arrive at your answers easily.

Example: $(\frac{3}{5}\cdot\frac{4}{7})+(\frac{2}{5}\cdot\frac{4}{7})=(\frac{3}{5}+\frac{2}{5})\cdot\frac{4}{7}=\frac{4}{7}$

[a] $(\frac{2}{3}\cdot\frac{5}{9})+(\frac{1}{3}\cdot\frac{5}{9})$

[b] $(\frac{7}{10}\cdot\frac{5}{12})+(\frac{3}{10}\cdot\frac{5}{12})$

[c] $(\frac{2}{3}\cdot\frac{5}{9})+(\frac{2}{3}\cdot\frac{4}{9})$

[d] $(\frac{7}{10}\cdot\frac{5}{12})+(\frac{7}{10}\cdot\frac{7}{12})$

[e] $(\frac{5}{6}\cdot\frac{7}{11})+(\frac{1}{6}\cdot\frac{7}{11})$

[f] $(\frac{5}{6}\cdot\frac{7}{11})+(\frac{5}{6}\cdot\frac{4}{11})$

[g] $(\frac{11}{13}\cdot\frac{15}{19})+(\frac{2}{13}\cdot\frac{15}{19})$

[h] $(\frac{11}{13}\cdot\frac{15}{19})+(\frac{11}{13}\cdot\frac{4}{19})$

[i] $(\frac{5}{3}\cdot\frac{3}{4})+(\frac{5}{3}\cdot\frac{1}{4})$

2. Find the products.

[a] $4\cdot6\frac{1}{4}$

[b] $7\cdot2\frac{1}{2}$

[c] $3\cdot4\frac{1}{6}$

[d] $\frac{1}{2}\cdot10\frac{1}{4}$

[e] $\frac{1}{3}\cdot15\frac{1}{2}$

[f] $\frac{3}{4}\cdot8\frac{2}{3}$

[g] $10\cdot8\frac{3}{10}$

[h] $\frac{2}{3}\cdot9\frac{3}{5}$

[i] $12\frac{1}{5}\times4$

[j] $8\frac{1}{4}\times\frac{3}{4}$

[k] $6\frac{1}{2}\times4\frac{1}{3}$

[l] $10\frac{1}{6}\times7\frac{1}{2}$

[m] $8\frac{1}{8}\times3\frac{1}{4}$

[n] $12\frac{1}{5}\times7\frac{1}{2}$

[o] $16\frac{3}{4}\times12\frac{1}{2}$

[p] $93\frac{1}{3}\times15\frac{2}{3}$

3. The example below shows how products are found by first replacing mixed numerals with improper fractions.

$$3\tfrac{1}{4}\cdot2\tfrac{1}{2}=8\tfrac{1}{8}$$
$$\downarrow\quad\downarrow\quad\uparrow$$
$$\tfrac{13}{4}\cdot\tfrac{5}{2}=\tfrac{65}{8}$$

Find the products as in the example.

[a] $1\frac{2}{3}\cdot3\frac{1}{2}$

[b] $1\frac{3}{4}\cdot1\frac{1}{2}$

[c] $2\frac{7}{8}\cdot1\frac{1}{4}$

[d] $6\frac{1}{3}\cdot5\frac{3}{4}$

[e] $8\frac{1}{5}\cdot6\frac{2}{3}$

[f] $3\frac{1}{3}\cdot3\frac{1}{3}$

[g] $12\frac{1}{2}\cdot2\frac{1}{2}$

[h] $3\frac{5}{6}\cdot2\frac{2}{3}$

[i] $1\frac{2}{7}\cdot4\frac{2}{3}$

[j] $16\frac{2}{3}\cdot1\frac{4}{5}$

★[k] $13\frac{3}{5}\cdot1\frac{3}{17}$

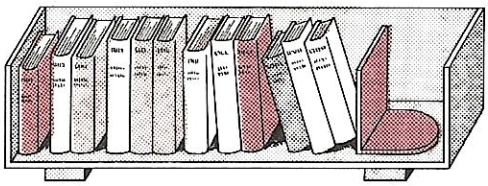
★[l] $20\frac{2}{5}\cdot8\frac{1}{3}$

Think

Find the smallest number ***y*** (between 100 and 125) so that the sum of all its factors is equal to $y+y+y$. (**Hint:** It must be a number with **many** factors.)

SHORT STORIES

1. Ate $\frac{3}{8}$ of $\frac{1}{2}$ of a cherry pie. How much did I eat?
2. Sally lives $\frac{3}{4}$ as far from school as Mary. Mary lives $\frac{1}{2}$ mi from school. How far does Sally live from school?
3. Jim is $\frac{5}{6}$ of Joe's age. Joe is 18. How old is Jim?
4. 12 books on the shelf. Each book $1\frac{3}{4}$ in. thick. What is the total thickness of the books?



5. A car averaged 60 mph for $2\frac{3}{4}$ hr. How far did it travel in this time?
6. Spent $\frac{1}{4}$ of my salary for food. Salary was $455.64. How much did I spend for food?
7. Multiplied a number by $\frac{2}{3}$. Multiplied the product by $\frac{3}{4}$. Could have gotten the same answer by multiplying by just one number. What number?
8. Bob's age is $\frac{5}{2}$ of Dave's age. Dave is 14. How old is Bob?
9. Spent $\frac{1}{2}$ of my money in one store. Then spent $\frac{1}{2}$ of my remaining money in another store. Started with $24. What do I have left?

★10. Spent $\frac{1}{2}$ of my money in one store. Spent $\frac{1}{3}$ of my remaining money in another store. Had $6.00 left. How much money at the start?

8-14 FINDING MISSING FACTORS

Finding missing factors in multiplication problems may help you find quotients in division problems.

Whole numbers	Rational numbers
F F P: $n \cdot 4 = 24$ ⟷ P F F: $24 \div 4 = n$ When you find the missing factor, → then you know the missing quotient.	F F P: $n \cdot \frac{2}{3} = \frac{4}{15}$ ⟷ P F F: $\frac{4}{15} \div \frac{2}{3} = n$ When you find the missing factor, → then you know the missing quotient.
Because $6 \cdot 4 = 24$, we know that $24 \div 4 = 6$	Because $\frac{2}{5} \cdot \frac{2}{3} = \frac{4}{15}$, we know that $\frac{4}{15} \div \frac{2}{3} = \frac{2}{5}$

Exercises

1. Find the missing factors.

[a] $6 = 6 \cdot n$ [b] $5 \cdot n = 1$ [c] $\frac{1}{2} \cdot n = 1$ [d] $\frac{3}{4} \cdot n = 1$ [e] $\frac{5}{2} \cdot n = 1$ [f] $\frac{7}{12} \cdot n = 1$ [g] $\frac{5}{8} \cdot n = \frac{5}{8}$ [h] $n \cdot \frac{3}{10} = \frac{3}{10}$

2. Find the missing factors.

[a] $n \cdot \frac{1}{5} = \frac{3}{5}$ [b] $6 \cdot n = \frac{6}{7}$ [c] $\frac{3}{2} \cdot n = \frac{3}{4}$ [d] $\frac{7}{4} \cdot n = \frac{7}{8}$ [e] $n \cdot \frac{1}{2} = \frac{6}{10}$ [f] $\frac{5}{2} \cdot n = 5$ [g] $\frac{3}{4} \cdot n = \frac{9}{16}$ [h] $\frac{1}{2} \cdot n = \frac{9}{16}$

3. Find the missing factor in the first equation. Then find the correct quotient for the screened equation.

[a] $n \cdot \frac{2}{3} = 1$ $1 \div \frac{2}{3} = n$

[b] $\frac{3}{10} \cdot n = 1$ $1 \div \frac{3}{10} = n$

[c] $n \cdot \frac{1}{8} = 1$ $1 \div \frac{1}{8} = n$

[d] $n \cdot \frac{1}{4} = \frac{1}{2}$ $\frac{1}{2} \div \frac{1}{4} = n$

[e] $n \cdot 1 = \frac{5}{6}$ $\frac{5}{6} \div 1 = n$

[f] $\frac{3}{5} \cdot n = \frac{9}{10}$ $\frac{9}{10} \div \frac{3}{5} = n$

[g] $\frac{1}{2} \cdot n = \frac{3}{8}$ $\frac{3}{8} \div \frac{1}{2} = n$

[h] $n \cdot \frac{3}{2} = \frac{9}{10}$ $\frac{9}{10} \div \frac{3}{2} = n$

[i] $n \cdot \frac{5}{8} = \frac{15}{32}$ $\frac{15}{32} \div \frac{5}{8} = n$

[j] $n \cdot \frac{4}{9} = \frac{8}{45}$ $\frac{8}{45} \div \frac{4}{9} = n$

[k] $\frac{3}{11} \cdot n = 1$ $1 \div \frac{3}{11} = n$

[l] $\frac{3}{5} \cdot n = \frac{21}{40}$ $\frac{21}{40} \div \frac{3}{5} = n$

4. Find the missing factor. Then write a division equation so that the quotient will be the missing factor of the multiplication equation.

[a] $\frac{3}{4} \cdot n = 1$ [b] $n \cdot \frac{5}{9} = 1$ [c] $\frac{3}{4} \cdot n = \frac{9}{20}$ [d] $n \cdot \frac{1}{2} = 4$ [e] $n \cdot \frac{1}{10} = \frac{1}{2}$ [f] $\frac{3}{5} \cdot n = \frac{3}{10}$ [g] $\frac{5}{7} \cdot n = \frac{25}{49}$ [h] $\frac{3}{8} \cdot n = \frac{9}{16}$ [i] $\frac{4}{5} \cdot n = \frac{12}{35}$ ★[j] $\frac{4}{7} \cdot n = \frac{2}{7}$ ★[k] $\frac{5}{12} \cdot n = \frac{10}{3}$ ★[l] $\frac{3}{2} \cdot n = \frac{2}{3}$

Think

Peasant Multiplication

27	and	35
54		17
108		8
216		4
432		2
864		1
945	=	27 · 35

(A) Choose two whole numbers.

(B) Double the first and halve the second, discarding any remainder.

(C) Continue.

(D) Continue.

(E) Continue.

(F) Continue.

(G) Now cross out all lines above that have even numbers for the second number. Add what remains in the left column.

Use Peasant Multiplication to find the products.

1. $59 \cdot 29$ **2.** $71 \cdot 57$ **3.** $101 \cdot 24$

8-15 DIVISION OF RATIONAL NUMBERS

The missing factor in Example A is $\frac{3}{2}$ because $\frac{3}{2} \cdot \frac{2}{3} = 1$.

The missing factor in Example B must be 5 times the missing factor in Example A because

$$5 \cdot \left(\frac{3}{2} \cdot \frac{2}{3}\right) = 5$$

We write: $5 \div \frac{2}{3} = 5 \cdot \frac{3}{2} = \frac{15}{2}$ or $7\frac{1}{2}$.

(A) $1 \div \frac{2}{3} =$ ▒ (P, F, F)

(B) $5 \div \frac{2}{3} =$ ▒ (P, F, F)

Discussion Exercises

1. What is the missing factor in C?
2. The missing factor in D must be ▒ times the missing factor in C.
3. Complete the statement: $4 \div \frac{3}{5} =$ ▒ $\cdot$ ▒ $=$ ▒.
4. The missing factor in E is ▒.
5. The missing factor in F must be ▒ times the missing factor in E.
6. Complete the statement: $\frac{1}{3} \div \frac{5}{8} =$ ▒ $\cdot$ ▒ $=$ ▒.

(C) $1 \div \frac{3}{5} =$ ▒ (P, F, F)

(D) $4 \div \frac{3}{5} =$ ▒ (P, F, F)

(E) $1 \div \frac{5}{8} =$ ▒ (P, F, F)

(F) $\frac{1}{3} \div \frac{5}{8} =$ ▒ (P, F, F)

Exercises

1. Find the number for a. Then find the quotient for b. Think of missing factors.

[a] $1 \div \frac{3}{8} = a \longrightarrow 2 \div \frac{3}{8} = b$

[b] $1 \div \frac{1}{2} = a \longrightarrow 5 \div \frac{1}{2} = b$

[c] $1 \div \frac{7}{5} = a \longrightarrow 3 \div \frac{7}{5} = b$

[d] $1 \div \frac{4}{3} = a \longrightarrow 8 \div \frac{4}{3} = b$

[e] $1 \div \frac{1}{7} = a \longrightarrow 6 \div \frac{1}{7} = b$

[f] $1 \div \frac{3}{2} = a \longrightarrow 4 \div \frac{3}{2} = b$

[g] $1 \div \frac{2}{5} = a \longrightarrow \frac{3}{4} \div \frac{2}{5} = b$

[h] $1 \div \frac{9}{10} = a \longrightarrow \frac{1}{5} \div \frac{9}{10} = b$

[i] $1 \div \frac{8}{3} = a \longrightarrow \frac{4}{3} \div \frac{8}{3} = b$

[j] $1 \div \frac{2}{5} = a \longrightarrow \frac{1}{2} \div \frac{2}{5} = b$

[k] $1 \div \frac{10}{7} = a \longrightarrow \frac{10}{7} \div \frac{10}{7} = b$

[l] $1 \div \frac{2}{7} = a \longrightarrow \frac{3}{5} \div \frac{2}{7} = b$

2. Solve the equations.

[a] $4 \div \frac{2}{3} = n \cdot \frac{3}{2}$

[b] $10 \div \frac{3}{8} = n \cdot \frac{8}{3}$

[c] $2 \div \frac{7}{11} = n \cdot \frac{11}{7}$

[d] $5 \div \frac{2}{5} = 5 \cdot n$

[e] $3 \div \frac{2}{3} = 3 \cdot n$

[f] $12 \div \frac{4}{3} = 12 \cdot n$

[g] $\frac{2}{3} \div \frac{2}{5} = \frac{2}{3} \cdot n$

[h] $\frac{5}{8} \div \frac{3}{4} = \frac{5}{8} \cdot n$

[i] $\frac{1}{2} \div \frac{2}{3} = \frac{1}{2} \cdot n$

[j] $\frac{4}{9} \div \frac{3}{4} = n \cdot \frac{4}{3}$

[k] $\frac{7}{2} \div \frac{5}{3} = n \cdot \frac{3}{5}$

[l] $\frac{8}{5} \div 2 = n \cdot \frac{1}{2}$

[m] $7 \div 5 = 7 \cdot n$

[n] $\frac{5}{6} \div \frac{5}{6} = \frac{5}{6} \cdot n$

★[o] $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot n$

3. Complete the sentence. To divide a first number by a second number, multiply the first number by the __?__ of the second number.

4. Find the quotients.

[a] $\frac{4}{5} \div \frac{2}{3}$

[b] $\frac{1}{6} \div \frac{2}{5}$

[c] $\frac{3}{7} \div \frac{1}{2}$

[d] $\frac{5}{12} \div \frac{5}{3}$

[e] $8 \div \frac{2}{3}$

[f] $\frac{4}{5} \div 2$

[g] $1 \div \frac{11}{12}$

[h] $\frac{9}{10} \div \frac{3}{2}$

[i] $\frac{25}{4} \div \frac{5}{8}$

[j] $\frac{7}{8} \div \frac{3}{4}$

[k] $\frac{3}{4} \div \frac{7}{8}$

[l] $3 \div \frac{1}{4}$

[m] $2 \div \frac{2}{3}$

[n] $6 \div \frac{1}{8}$

[o] $10 \div \frac{3}{8}$

[p] $8 \div \frac{5}{9}$

[q] $5 \div \frac{3}{5}$

[r] $\frac{1}{2} \div \frac{1}{4}$

[s] $\frac{1}{2} \div \frac{3}{4}$

[t] $\frac{4}{5} \div \frac{1}{2}$

[u] $\frac{7}{8} \div \frac{1}{3}$

[v] $\frac{3}{10} \div \frac{1}{10}$

[w] $\frac{4}{9} \div \frac{2}{3}$

[x] $\frac{20}{3} \div 5$

[y] $\frac{5}{8} \div \frac{5}{4}$

[z] $\frac{7}{10} \div \frac{3}{5}$

★**5.** If $n \div \frac{4}{9} = \frac{4}{9}$, then which number must n be?

[a] 1 [b] $\frac{4}{9}$ [c] $\frac{9}{4}$ [d] $\frac{16}{81}$ [e] $\frac{81}{16}$

8-16 WHOLE NUMBERS, RATIONAL NUMBERS, AND DIVISION

We may establish certain relationships between whole numbers and rational numbers in division problems.

A We can find the quotient for $34 \div 9$ by writing

$$34 \div 9 = \frac{34}{1} \div \frac{9}{1} = \frac{34}{1} \cdot \frac{1}{9} = \frac{34}{9}$$

B The improper fraction $\frac{34}{9}$ may be written as a mixed numeral through the division algorithm

$$\begin{array}{r} 3, \text{R}7 \\ 9\overline{)34} \end{array} \longrightarrow \frac{34}{9} = \frac{3 \cdot 9}{9} + \frac{7}{9} \longrightarrow \frac{34}{9} = 3\frac{7}{9}$$

Let us agree to write $\frac{26}{5} \div \frac{13}{3}$ as $\dfrac{\frac{26}{5}}{\frac{13}{3}}$.

We call this a **complex fraction** and say that $\frac{26}{5}$ is its numerator and $\frac{13}{3}$ is its denominator.

Discussion Exercises

1. Explain how you arrive at the final result $6\frac{3}{7}$.

$$\begin{array}{r} 6, \text{R}3 \\ 7\overline{)45} \end{array} \longrightarrow \begin{array}{r} 6\frac{3}{7} \\ 7\overline{)45} \end{array}$$

The following exercises illustrate methods for simplifying complex fractions. Complete each equation and explain each step.

2. [a] $\dfrac{\frac{26}{5}}{\frac{13}{3}} = \frac{26}{5} \div \frac{13}{3} = \frac{26}{5} \cdot \frac{3}{13} = ?$

[b] $\dfrac{\frac{26}{5}}{\frac{13}{3}} = \dfrac{\frac{26}{5} \cdot \frac{3}{13}}{\frac{13}{3} \cdot \frac{3}{13}} = \frac{6}{5} = ?$

3. [a] $\dfrac{\frac{3}{5}}{\frac{2}{7}} = \frac{3}{5} \div \frac{2}{7} = \frac{3}{5} \cdot \frac{7}{2} = ?$

[b] $\dfrac{\frac{3}{5}}{\frac{2}{7}} = \dfrac{\frac{3}{5} \cdot \frac{7}{2}}{\frac{2}{7} \cdot \frac{7}{2}} = \frac{21}{10} = ?$

Exercises

1. Give each quotient in the form $\frac{a}{b}$.

[a] $4 \div 5$
[b] $7 \div 16$
[c] $8 \div 7$
[d] $11 \div 3$
[e] $19 \div 25$
[f] $23 \div 4$
[g] $17 \div 6$
[h] $42 \div 9$
[i] $13 \div 26$
[j] $\frac{2}{3} \div \frac{1}{5}$
[k] $\frac{3}{7} \div \frac{7}{3}$
[l] $\frac{5}{9} \div \frac{10}{19}$

2. Give each quotient as a mixed numeral.

[a] $17 \div 4$
[b] $38 \div 5$
[c] $37 \div 12$
[d] $101 \div 10$
[e] $33 \div 2$
[f] $100 \div 3$
[g] $79 \div 15$
[h] $332 \div 5$
[i] $2137 \div 10$
[j] $348 \div 21$
[k] $1622 \div 30$
[l] $5155 \div 42$

3. We have agreed that the symbol

$$\frac{\frac{2}{3}}{\frac{5}{7}} \text{ means } \frac{2}{3} \div \frac{5}{7}$$

Find the quotients in the following exercises.

[a] $\dfrac{\frac{1}{2}}{\frac{3}{8}}$
[b] $\dfrac{\frac{5}{9}}{\frac{1}{3}}$
[c] $\dfrac{\frac{7}{6}}{\frac{7}{3}}$
[d] $\dfrac{\frac{3}{4}}{\frac{9}{4}}$
[e] $\dfrac{4\frac{1}{2}}{9}$
[f] $\dfrac{8\frac{1}{3}}{25}$
[g] $\dfrac{37\frac{1}{2}}{100}$
[h] $\dfrac{3\frac{1}{3}}{13\frac{1}{3}}$
[i] $\dfrac{2\frac{1}{2}}{10}$
[j] $\dfrac{7}{4\frac{2}{3}}$
[k] $\dfrac{4\frac{2}{5}}{2\frac{3}{4}}$
[l] $\dfrac{2\frac{7}{8}}{3\frac{5}{6}}$

4. Use improper fractions to help you find the quotients.

Example: $3\frac{3}{4} \div 1\frac{2}{3} = \frac{15}{4} \div \frac{5}{3} = \frac{\cancel{15}^{3}}{4} \cdot \frac{3}{\cancel{5}_{1}} = \frac{9}{4}$

[a] $3\frac{1}{2} \div 1\frac{2}{5}$
[b] $2\frac{3}{4} \div 4\frac{1}{2}$
[c] $3\frac{1}{5} \div 1\frac{1}{7}$
[d] $4\frac{2}{3} \div 1\frac{3}{4}$
[e] $5\frac{3}{5} \div 4\frac{2}{3}$
[f] $2\frac{1}{3} \div 5\frac{1}{4}$
[g] $12\frac{1}{2} \div 5$
[h] $14 \div 4\frac{2}{3}$
[i] $7\frac{1}{6} \div 14\frac{1}{3}$
[j] $6\frac{2}{3} \div \frac{1}{3}$
[k] $1\frac{7}{8} \div 1\frac{1}{4}$
[l] $1\frac{4}{5} \div 2\frac{7}{10}$

5. Solve the equations.

[a] $0 \div \frac{3}{4} = n$
[b] $n \div \frac{3}{4} = 0$
[c] $1 \div \frac{3}{4} = n$
[d] $n \div \frac{3}{4} = 1$
[e] $\frac{3}{4} \div n = 1$
[f] $n \div \frac{3}{4} = \frac{3}{4}$
[g] $\frac{3}{4} \div n = \frac{3}{4}$
[h] $\frac{3}{4} \div \frac{4}{3} = n$
[i] $\frac{3}{4} \div \frac{3}{4} = n$
[j] $\frac{3}{4} \div n = \frac{4}{3}$
[k] $\frac{4}{3} \div n = \frac{3}{4}$
[l] $\frac{3}{4} \div 1 = n$
[m] $\left(\frac{3}{4} \div \frac{3}{4}\right) \div n = \frac{3}{4}$
[n] $\frac{3}{4} \div \left(\frac{3}{4} \div n\right) = \frac{3}{4}$

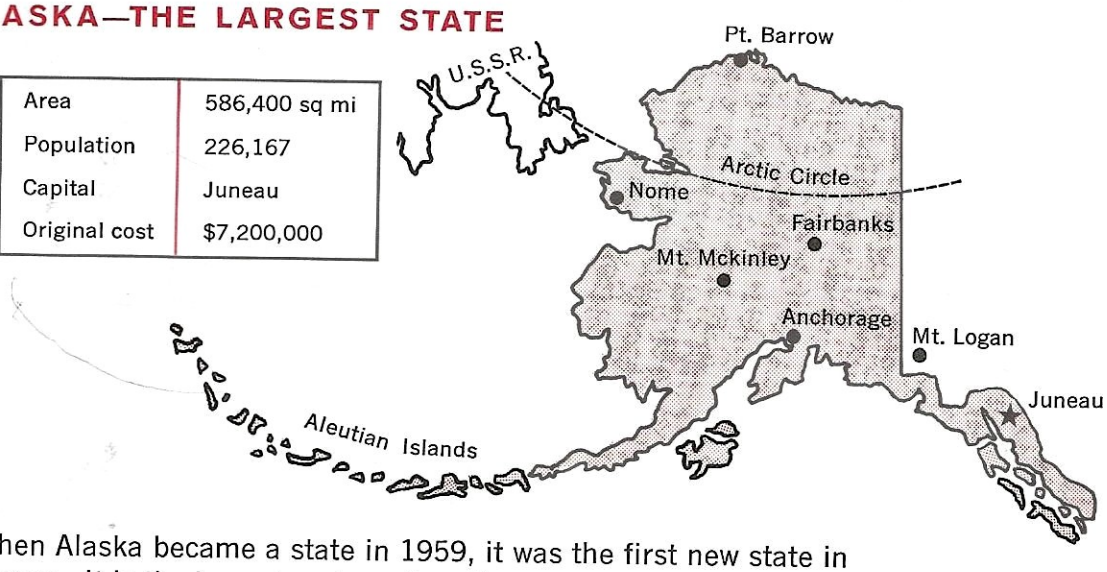
Think

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

In the equation, a, b, and c are different whole number denominators. What are these denominators?

ALASKA—THE LARGEST STATE

Area	586,400 sq mi
Population	226,167
Capital	Juneau
Original cost	$7,200,000



When Alaska became a state in 1959, it was the first new state in 47 years. It is the largest and most northern of all the United States. While we may think of Alaska as being near the north pole, the most northern point of Alaska is still over 1200 miles south of the north pole.

Exercises

Use the preceding table to answer these questions.

1. The area of Alaska is about $\frac{1}{6}$ of the area of all 50 of the United States. Use this fact to estimate the area of the United States. Round your answer to the nearest 100,000 sq mi.

2. The population of Alaska was about $\frac{1}{80}$ of the population of California in 1960. About how many people lived in California in 1960?

3. There were 18,531 more people living in Wichita, Kansas, in 1960 than in all of Alaska. What was the population of Wichita?

4. The area of Alaska is very nearly 483 times the area of Rhode Island. Estimate the area of Rhode Island, rounding to the nearest square mile.

5. The population of Rhode Island in 1960 was 45,180 less than four times the population of Alaska. What was the population of Rhode Island in 1960?

6. The area of Alaska is 51,722 sq mi more than twice the size of the state of Texas. What is the area of Texas?

7. About $\frac{1}{3}$ of the area of Alaska lies inside the arctic circle. How many square miles is this?

8. Mount McKinley, in Alaska, is the highest mountain in North America. Mount Logan, which is near the Alaskan border, is the second highest mountain. Mount Logan is 19,850 ft high and McKinley is 470 ft higher. How high is McKinley?

9. Mount Everest, in the Himalayan mountains, is the highest mountain in the world. It is 29,028 ft high. How much higher than McKinley is Everest?

10. There are 640 acres in a square mile. How many acres are in Alaska?

Mount McKinley

★**11.** The United States purchased the territory of Alaska from Russia in 1867 for $7,200,000. Find the cost of Alaska to the nearest cent per acre.

8-17 PROBLEM SOLVING—DIVISION

Most of the problems in this section name just two numbers. You can get each answer by dividing one number by the other. Write the correct division equation for each exercise and then find the solution of the equation. In some exercises, it will be helpful to first write a multiplication equation. The first exercise is worked as an example.

Exercises

1. $\frac{3}{4}$ of Jim's age is 12. How old is Jim?
Answer: Call Jim's age n

$$\frac{3}{4} \cdot n = 12$$
$$n = 12 \div \frac{3}{4} = 12 \cdot \frac{4}{3}$$
$$n = 16$$

2. $\frac{2}{3}$ of a number is $\frac{16}{3}$. What is the number?

3. $\frac{2}{3}$ of a number is $\frac{3}{4}$. What is the number?

4. Harry's height is $\frac{7}{8}$ of Paul's height. Harry is 63 in. tall. How tall is Paul?

5. A plane flying 636 mph is flying at about $\frac{6}{7}$ the speed of sound at sea level. What is the speed of sound at sea level?

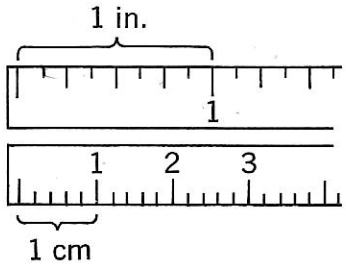
6. Joe walked $8\frac{1}{8}$ mi in $2\frac{1}{2}$ hr. What was his average walking speed in miles per hour?

7. A car traveled 98 mi in $1\frac{3}{4}$ hr. What was the average speed of the car?

8. A kilometer is about $\frac{5}{8}$ mi. How many kilometers in $\frac{3}{4}$ mi?

9. A square foot is $\frac{1}{9}$ of a sq yd. How many square feet in $3\frac{1}{2}$ sq yd?

10. A centimeter is about $\frac{2}{5}$ in. About how many centimeters are there in 12 in.



11. A watch cost $\frac{3}{5}$ as much as a ring. The watch cost \$60. What did the ring cost?

12. The product of two numbers is $\frac{8}{3}$. One of the numbers is $\frac{1}{3}$. Find the other number.

★ 13. The quotient of two numbers is $\frac{8}{3}$. The larger number is $\frac{1}{3}$. Find the other number.

★ 14. The product of two numbers is $\frac{2}{3}$. One of the numbers is $\frac{5}{4}$. Find the other number.

★ 15. The quotient of two numbers is $\frac{2}{3}$. The smaller number is $\frac{5}{4}$. Find the other number.

★ 16. A carpenter needs to divide a board into five pieces of equal length. If the board is $42\frac{3}{4}$ in. long and a width of $\frac{1}{16}$ in. is lost by each saw cut, how long will each piece be?

Think

Check to see if each statement in the sets given in Exercises 1 through 4 is correct. If so, find two more statements of the same kind in that set.

1. $1\frac{1}{2} \cdot 3 = 1\frac{1}{2} + 3$
   $1\frac{1}{3} \cdot 4 = 1\frac{1}{3} + 4$
   $1\frac{1}{4} \cdot 5 = 1\frac{1}{4} + 5$

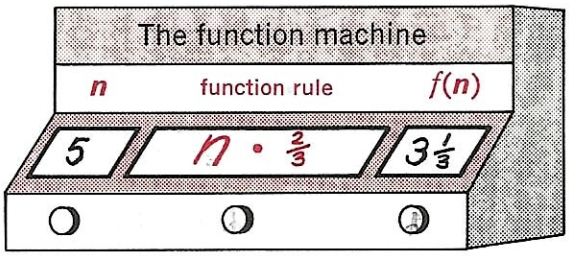
2. $1 \cdot \frac{1}{2} = 1 - \frac{1}{2}$
   $2 \cdot \frac{2}{3} = 2 - \frac{2}{3}$
   $3 \cdot \frac{3}{4} = 3 - \frac{3}{4}$

3. $1\frac{1}{3} \div \frac{2}{3} = 1\frac{1}{3} + \frac{2}{3}$
   $2\frac{1}{4} \div \frac{3}{4} = 2\frac{1}{4} + \frac{3}{4}$
   $3\frac{1}{5} \div \frac{4}{5} = 3\frac{1}{5} + \frac{4}{5}$

4. $4\frac{1}{2} \div 3 = 4\frac{1}{2} - 3$
   $5\frac{1}{3} \div 4 = 5\frac{1}{3} - 4$
   $6\frac{1}{4} \div 5 = 6\frac{1}{4} - 5$

8-18 RATIONAL NUMBER FUNCTIONS

The function machine shows the function rule $f(n) = n \cdot \frac{2}{3}$. When the input number is 5, the output number is $5 \cdot \frac{2}{3}$ or $3\frac{1}{3}$. We write, $f(5) = 3\frac{1}{3}$.



Exercises

1. Use the function rule $f(n) = n \cdot \frac{2}{3}$ to complete each exercise.

[a] $f(2) =$ [d] $f(0) =$ [g] $f(3\frac{1}{2}) =$ [j] $f(\quad) = 2\frac{1}{5}$

[b] $f(3) =$ [e] $f(\frac{3}{2}) =$ [h] $f(\quad) = 1$ [k] $f(\frac{2}{3}) =$

[c] $f(1) =$ [f] $f(1\frac{1}{5}) =$ [i] $f(\quad) = \frac{3}{2}$ [l] $f(\quad) = 100$

2. Suppose a function rule is to multiply the input n by $5\frac{1}{2}$. If you know that your output number is $7\frac{1}{3}$, how do you find n?

3. Suppose a function rule requires dividing the input n by $3\frac{1}{3}$. If you know the output number is $\frac{3}{5}$, how do you find the input number? What number is it?

In Exercises 4 through 12, give what should go in each screen.

4.

function rule $n \cdot \frac{1}{4}$		
	n	$f(n)$
[a]	4	
[b]	$\frac{2}{5}$	
[c]	1	
[d]	$\frac{5}{4}$	
[e]	$1\frac{2}{3}$	

5.

function rule $\frac{2}{5} \cdot n$		
	n	$f(n)$
[a]	$\frac{1}{5}$	
[b]	6	
[c]		$3\frac{1}{2}$
[d]		1
[e]	$2\frac{3}{5}$	

6.

function rule $n \cdot 1\frac{2}{3}$		
	n	$f(n)$
[a]	0	
[b]	$\frac{2}{7}$	
[c]		$2\frac{2}{5}$
[d]		2
[e]	$3\frac{3}{4}$	

7.

function rule: $\left(\frac{3}{2} \cdot n\right) + \frac{1}{2}$

	n	$f(n)$
[a]	4	
[b]	$\frac{7}{5}$	
[c]		$\frac{1}{2}$
[d]		$\frac{3}{2}$
[e]	$4\frac{1}{7}$	

8.

[a] function rule: $n \cdot$ ____

n	$f(n)$
$\frac{1}{3}$	$\frac{1}{6}$
$\frac{2}{5}$	$\frac{1}{5}$
4	2
$\frac{8}{3}$	$\frac{4}{3}$
[b] $2\frac{3}{5}$	

9.

function rule: $n \div \frac{1}{2}$

	n	$f(n)$
	$\frac{2}{3}$	$\frac{4}{3}$
[a]	$\frac{3}{5}$	
[b]	3	
[c]	$3\frac{2}{7}$	
[d]	$4\frac{1}{5}$	

10.

function rule: $n \div 1\frac{1}{3}$

	n	$f(n)$
[a]	2	
[b]	$\frac{2}{3}$	
[c]	5	
[d]		1
[e]		$\frac{8}{3}$

★11.

[a] function rule: $n \div$ ____

n	$f(n)$
3	$\frac{4}{3}$
$3\frac{1}{4}$	$\frac{13}{9}$
$\frac{2}{5}$	$\frac{8}{45}$
$\frac{18}{11}$	$\frac{8}{11}$
[b] $\frac{6}{7}$	

★12.

[a] function rule: ____

n	$f(n)$
$\frac{5}{6}$	1
$\frac{4}{7}$	$\frac{24}{35}$
10	12
[b] $2\frac{2}{3}$	
[c]	$\frac{27}{40}$

13. The amount of postage needed for letters and packages can be shown by using the function machine. For example, we can think of first-class postage and find output numbers as in the following tables. Note that a letter $\frac{7}{8}$ oz costs 5¢ to mail and a letter $1\frac{1}{8}$ oz costs 10¢ to mail. Study and complete the tables. The first table is completed for you.

function rule: first-class postage

n	$f(n)$
$\frac{1}{2}$	5
$\frac{7}{8}$	5
$1\frac{1}{8}$	10
$1\frac{15}{16}$	10
$2\frac{3}{8}$	15
$\frac{9}{2}$	25
6	30

function rule: first-class postage

	n	$f(n)$
[a]	$\frac{1}{3}$	
[b]	1	
[c]	$2\frac{1}{16}$	
[d]	$2\frac{15}{16}$	
[e]	$4\frac{7}{16}$	
[f]	$4\frac{3}{4}$	
[g]	$\frac{41}{8}$	

function rule: first-class postage

	n	$f(n)$
[h]	$\frac{3}{2}$	
[i]	$\frac{9}{4}$	
[j]	$3\frac{5}{8}$	
[k]	$5\frac{11}{16}$	
[l]	7	
[m]	$\frac{33}{4}$	
[n]	$\frac{161}{16}$	

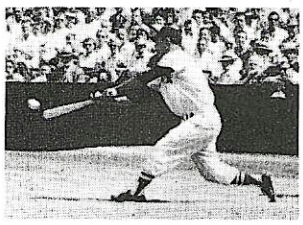
★ **14.** Graph the numbers in the first-class postage function tables.

★ **15.** Suppose you know that the output number for the first-class postage function is 15.

[a] What is the largest number that n may be?

[b] The number for n must be more than what number?

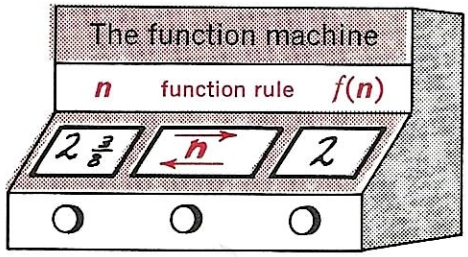
Think



1. Find the average of the five numbers: 0, 1, 0, 0, 1.
2. How is the Problem 1 related to finding the batting average of a baseball player who goes "2 for 5" in the first game of the season?

A SPECIAL FUNCTION

The input and output numbers in the tables describe a special kind of function rule. We use the symbol $\overrightarrow{n}$ for this function rule. See if you can discover what this rule is.



A

function rule $\overrightarrow{n}$

n	$f(n)$
2	2
$2\frac{1}{8}$	2
$2\frac{1}{4}$	2
$2\frac{3}{8}$	2
$2\frac{1}{2}$	3
$2\frac{3}{4}$	3

B

function rule $\overrightarrow{n}$

n	$f(n)$
4	4
$4\frac{1}{5}$	4
$4\frac{2}{5}$	4
$4\frac{3}{5}$	5
$4\frac{4}{5}$	5
5	5

C

function rule $\overrightarrow{n}$

n	$f(n)$
$\frac{5}{6}$	1
$\frac{7}{4}$	2
$\frac{1}{8}$	0
$\frac{15}{4}$	4
$8\frac{7}{12}$	9
$16\frac{1}{2}$	17

Exercises

For Exercises 1 through 3, give the output number $f(n)$ for each input number n.

1.

function rule $\overrightarrow{n}$

	n	$f(n)$
[a]	1	
[b]	$\frac{4}{3}$	
[c]	$\frac{5}{3}$	
[d]	2	
[e]	$\frac{7}{3}$	
[f]	$\frac{11}{3}$	

2.

function rule $\overrightarrow{n}$

	n	$f(n)$
[a]	$\frac{9}{10}$	
[b]	$\frac{17}{16}$	
[c]	$\frac{20}{6}$	
[d]	$\frac{44}{9}$	
[e]	$\frac{100}{3}$	
[f]	$\frac{1}{6}$	

3.

function rule $\overrightarrow{n}$

	n	$f(n)$
[a]	$18\frac{5}{8}$	
[b]	$9\frac{4}{11}$	
[c]	$\frac{237}{241}$	
[d]	$\frac{289}{144}$	
[e]	$1099\frac{9}{10}$	
[f]	$\frac{11}{24}$	

You have probably discovered that the function rule can be thought of as finding the "nearest whole number." If we think of the function in this way, we must make an agreement about rational numbers halfway between two whole numbers. As we have seen, for input numbers like $4\frac{1}{2}$, the output number is 5.

Study the examples below before trying the remaining exercises.

(A) $\overrightarrow{\frac{9}{2}} = 5$ (B) $\overrightarrow{2\frac{1}{3}} + \overrightarrow{3\frac{1}{3}} = 5$

4. Give the missing numbers.

[a] $\overrightarrow{\frac{8}{3}} = ?$ [b] $\overrightarrow{\frac{15}{16}} = ?$ [c] $\overrightarrow{7\frac{3}{4}} = ?$ [d] $\overrightarrow{\frac{541}{100}} = ?$ [e] $\overrightarrow{\frac{289}{99}} = ?$ [f] $\overrightarrow{\frac{38}{7}} = ?$

5. Solve the equations.

[a] $\overrightarrow{1\frac{3}{8}} + \overrightarrow{2\frac{1}{4}} = n$

[b] $\overrightarrow{1\frac{3}{8} + 2\frac{1}{4}} = n$

[c] $\overrightarrow{8\frac{3}{4}} - \overrightarrow{3\frac{3}{8}} = n$

[d] $\overrightarrow{8\frac{3}{4} - 3\frac{3}{8}} = n$

[e] $\overrightarrow{2\frac{1}{3}} \cdot \overrightarrow{3\frac{3}{4}} = n$

[f] $\overrightarrow{2\frac{1}{3} \cdot 3\frac{3}{4}} = n$

[g] $\overrightarrow{11\frac{1}{2}} \cdot \overrightarrow{2\frac{1}{2}} = n$

★[h] $\overrightarrow{2\frac{1}{2} \cdot 24\frac{1}{4}} = n$

★[i] $2\frac{1}{2} \cdot \overrightarrow{24\frac{1}{4}} = n$

6. You can find many numbers for N in the equation.

$$2\tfrac{2}{3} \cdot N = 6$$

[a] What is the smallest number for N?
[b] The number for N must be less than ▮.
[c] Is there a largest number for N?

Think

Discover the rule for the function shown in the table.

function rule

n	$f(n)$
$\frac{3}{4}$	1
$2\frac{1}{7}$	10
$2\frac{6}{7}$	11
$5\frac{7}{8}$	110
$\frac{46}{3}$	1111
$\frac{31}{2}$	10000
$13\frac{2}{5}$	1101

8-19 PROBLEMS—RATIONAL NUMBERS

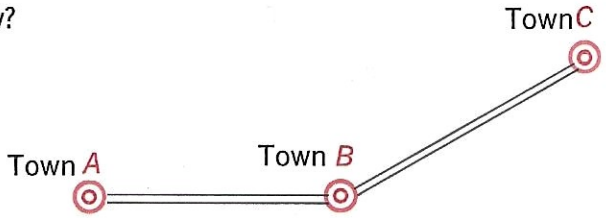
Exercises

For each exercise, give estimates using whole numbers. Then find the correct answer.

1. $\frac{9}{16}$ mi to school. Walked to school and back for 20 days. How far?

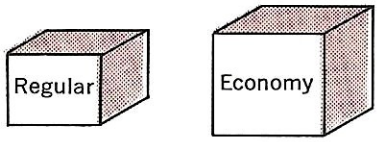
2. $53\frac{7}{8}$ in. tall. Grew $2\frac{7}{8}$ in. How tall now?

3. $29\frac{3}{4}$ mi from Town A to Town B. It is 63 mi from Town A to Town C, going through B. How far from Town B to Town C?



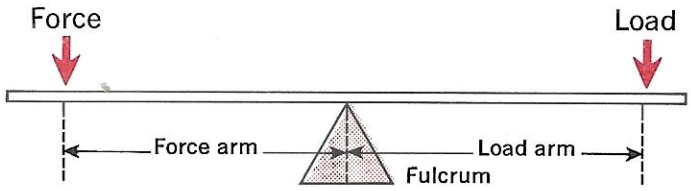
4. Averaged $18\frac{9}{10}$ mi per gal of gasoline on a trip. Used $21\frac{3}{10}$ gal of gasoline. How far did I travel?

5. Economy size: $63\frac{2}{3}$ grams (gm).
Regular size: $38\frac{3}{4}$ gm.
How much more does economy size weigh?

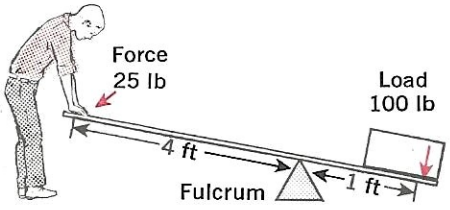


6. Gasoline costs $33\frac{9}{10}¢$ per gallon. Bought 10 gal. What is the cost of the gasoline?
7. 440-yd dash. Time: $61\frac{1}{10}$ sec. How many yards per second?
8. Piece of wire: $26\frac{1}{4}$ in. long. Cut into pieces $3\frac{3}{4}$ in. long. How many pieces of wire?
9. Walked at a rate of $3\frac{3}{4}$ mph. Speed of car is $10\frac{1}{4}$ times as fast. How fast is the car?
10. Rectangle: length $10\frac{1}{4}$ in., width $3\frac{7}{8}$ in. What is the perimeter?
11. Coffee: 89¢ per lb. 1 lb makes 44 cups. Cost of coffee per cup?
12. Satellite travels 42,000 mi in $2\frac{2}{5}$ hr. What is the speed of the satellite?
13. Yard goods: 79¢ per yard. Bought $2\frac{1}{4}$ yd. What did I pay?
14. Rocket speed: $7\frac{1}{3}$ mi per sec. How fast in miles per hour?

LEVERS

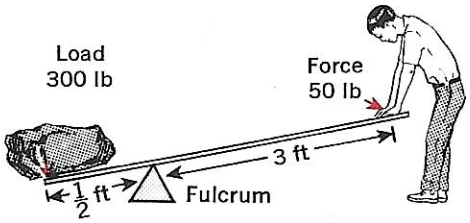


Levers are used to help move heavy objects with a small amount of force. The **balance point** (pivot point) of a lever is called the **fulcrum**. Study the drawings to help you understand the law of simple levers.



A force of 25 lb must be applied 4 ft from the fulcrum to balance a 100-lb load which is 1 ft from the fulcrum.

$25 \cdot 4 = 100 \cdot 1$



A force of 50 lb must be applied 3 ft from the fulcrum to balance a 300-lb load which is $\frac{1}{2}$ ft from the fulcrum.

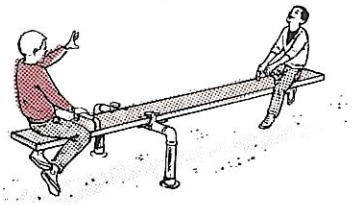
$50 \cdot 3 = \frac{1}{2} \cdot 300$

Exercises

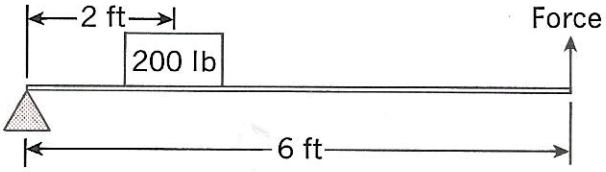
1. Give the number for the missing part for each line of the table.

	Load	Length of load arm	Force	Length of force arm
[a]	200 lb	2 ft		4 ft
[b]	600 lb	$\frac{1}{3}$ ft	100 lb	
[c]	1000 lb		50 lb	10 ft
[d]		$\frac{3}{4}$ ft	20 lb	6 ft
[e]	720 lb	1 in.	90 lb	
[f]	4000 lb		100 lb	3 ft

2. Joe weighs 120 lb and sits 3 ft from the balance point (fulcrum) of a seesaw. Tom weighs 80 lb and balances with Joe. How far is Tom from the balance point?



★ **3.** Find the force required to lift a 200-lb load if the force is applied upward at the end of the lever as shown at the right.



Think

Fill in the missing squares so that each row, each column, and the two diagonals add up to the same rational number for each of A and B.

A

$\frac{1}{2}$		$\frac{1}{6}$
$\frac{1}{12}$	$\frac{5}{12}$	
$\frac{2}{3}$		$\frac{1}{3}$

B

$6\frac{1}{4}$		$11\frac{1}{4}$
	$7\frac{1}{2}$	
		$8\frac{3}{4}$

CHAPTER REVIEW

Key Concepts

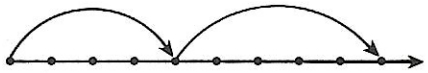
1. Addition and subtraction of rational numbers may be visualized with the aid of the number line.
2. You can easily find sums of rational numbers if you choose fractions having the same denominators: in symbols,
$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$$
3. The **least common denominator** for two fractions is the **least common multiple** of their denominators.
4. Both improper fractions and mixed numerals are used to represent rational numbers.
5. The basic principles play an important role in computing with rational numbers.
6. You should understand how to multiply rational numbers:
$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$$
7. If the product of two rational numbers is 1, then each number is the **reciprocal** of the other.
8. To divide a first rational number by a second, multiply the first rational number by the reciprocal of the second:
$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$$
9. If a and b are whole numbers ($b\neq 0$), then $a\div b$ is equal to the rational number $\frac{a}{b}$.

Vocabulary List

associative principle
commutative principle
denominator
distributive principle
equivalent fractions
factor
function
graph
identity principle:
  one
  zero
improper fraction
least common denominator (LCD)
mixed numeral
number line
number pairs
numerator
prime factorization
reciprocal
reducing
region

Review Exercises

1. Tell what addition fact is shown by the number line, given that it is scaled in
[a] units
[b] halves
[c] ninths
[d] twenty-fourths



2. Give the number for n.
[a] $\frac{3}{5}+n=1$
[b] $\frac{5}{7}+n=1\frac{3}{7}$
[c] $1\frac{1}{3}+n=2$
[d] $\frac{7}{9}-n=\frac{4}{9}$
[e] $4\frac{1}{3}-n=3\frac{2}{3}$
[f] $n-\frac{3}{4}=\frac{5}{4}$
[g] $n-\frac{11}{7}=\frac{12}{7}$
[h] $\frac{4}{3}\cdot n=\frac{28}{3}$

3. Add and reduce to lowest terms.
[a] $\frac{1}{6}+\frac{1}{10}$
[b] $\frac{2}{15}+\frac{2}{21}$

4. Use the associative and commutative principles to obtain the sums easily.
[a] $2\frac{1}{2}+\frac{3}{5}+1\frac{1}{2}$
[b] $\frac{7}{9}+3\frac{2}{5}+\frac{2}{9}$
[c] $\frac{7}{13}+\frac{5}{11}+\frac{6}{13}$
[d] $2\frac{3}{5}+3\frac{4}{7}+5\frac{2}{5}$

5. Use the distributive principle to obtain the answers quickly.
[a] $\left(\frac{3}{5}\cdot\frac{7}{9}\right)+\left(\frac{2}{5}\cdot\frac{7}{9}\right)$
[b] $\left(\frac{11}{17}\cdot\frac{12}{19}\right)+\left(\frac{6}{17}\cdot\frac{12}{19}\right)$

6. Find the products.

[a] $3 \cdot 2\frac{2}{3}$ [c] $1\frac{1}{2} \cdot 6$ [e] $\frac{13}{5} \cdot 1\frac{2}{3}$

[b] $8 \cdot 5\frac{1}{4}$ [d] $2\frac{1}{3} \cdot 3\frac{1}{2}$ [f] $2\frac{2}{5} \cdot 4\frac{1}{6}$

7. Solve the equations.

[a] $\left(\frac{4}{3} \cdot n\right) \cdot \frac{5}{7} = \frac{5}{7}$

[b] $\left(\frac{2}{5} \cdot n\right) \cdot \frac{4}{9} = \frac{2}{5}$

[c] $\frac{3}{5} \cdot \frac{2}{7} \cdot n = 1$

[d] $\left(n + \frac{3}{4}\right) \cdot \frac{4}{5} = 1$

8. Give the reciprocal of

[a] $\frac{11}{13}$ [c] $\frac{17}{3}$ [e] $\frac{2}{3} \cdot \frac{5}{7}$

[b] $3\frac{1}{2}$ [d] $4\frac{3}{7}$ [f] $\frac{3}{5} - \frac{1}{2}$

9. Find the quotients.

[a] $\frac{3}{5} \div \frac{3}{8}$ [e] $4\frac{2}{3} \div 1\frac{3}{4}$

[b] $\frac{5}{7} \div \frac{15}{14}$ [f] $5\frac{3}{5} \div 4\frac{2}{3}$

[c] $\frac{24}{17} \div \frac{12}{7}$ [g] $\dfrac{\frac{3}{4}}{\frac{3}{8}}$

[d] $3\frac{1}{2} \div 1\frac{2}{5}$ [h] $\dfrac{1\frac{1}{4}}{3\frac{1}{2}}$

10. A carpenter saws $2\frac{3}{8}$ in. from one end of a board $21\frac{1}{2}$ in. long. How long is the remaining piece?

11. Express 45 min as a fraction of 1 hr.

12. How many books $1\frac{3}{4}$ in. thick can be placed on a shelf 21 in. wide? On a shelf 22 in. wide?

13. A car drove 60 mph for 1 hr 45 min. How far did it go?

14. A car drove 50 mph for 1 hr 45 min. How far did it go?

15. A dealer paid \$2400 for a car. He added on $\frac{1}{5}$ of this cost in order to determine his selling price. What was his selling price?

16. The width of a rectangle is $\frac{7}{8}$ in. Find the perimeter, given that the length is $\frac{1}{2}$ in. more than the width.

17. A man spent $\frac{3}{5}$ of his money for a watch. If the watch cost \$60, how much money had he originally?

18. Two circles are drawn on a sheet of paper. The distance between their centers is $4\frac{1}{2}$ in. One circle has a diameter of $1\frac{1}{2}$ in. and the other has a diameter of $\frac{3}{4}$ in. Find the distance between the two points, one on each circle, which are closer together than any other such pair of points.

19. The perimeter of a rectangle is 7 in. and the width is $1\frac{3}{4}$ in. Find the length.

20. Two sides of a triangle are $\frac{5}{3}$ and $\frac{7}{4}$ in. long. The perimeter is 5 in. Find the length of the third side.

21. The base of an isosceles triangle is $\frac{3}{4}$ in. long. If the perimeter is $3\frac{1}{2}$ in., how long is each of the other two sides?

CHAPTER SUPPLEMENT

Reading List

GARDNER, MARTIN. *The Second Scientific American Book of Puzzles and Diversions.* New York, Simon and Schuster, 1961; pp. 104–111.

HOGBEN, LANCELOT. *Mathematics in the Making.* Garden City, New York, Doubleday and Company, 1960; pp. 50–55.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS. *Twenty-seventh Yearbook.* Washington, D.C., 1963; pp. 245–260.

SMITH, D. E. *History of Mathematics.* New York. Dover Publications, 1953; Vol. II, Chap. 4, pp. 208–228.

STICKER, HENRY. *How to Calculate Quickly.* New York, Dover Publications, 1955; pp. 131–171.

VAN DER WAERDEN, B. L. *Science Awakening.* New York, John Wiley and Sons, 1963; pp. 24–36.

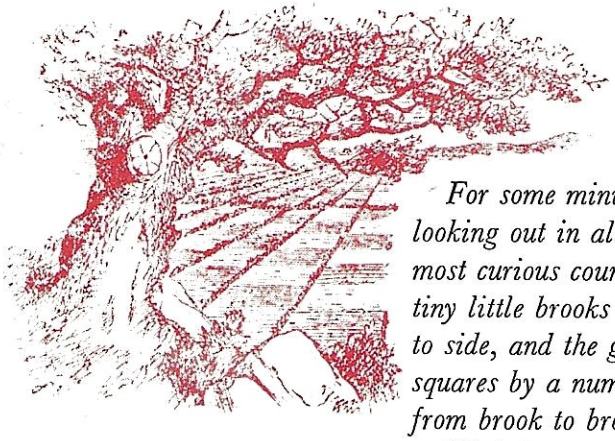
Activities

A. Read about Diophantine brain teasers. The term **Diophantine** is descended from Diophantus of Alexandria who was the first Greek algebraist to clearly recognize fractions as numbers. (See *The Second Scientific American Book of Puzzles and Diversions*, pp. 104–111.)

B. Choose a stock from one of the stock exchanges. Follow its "ups and downs" for two weeks, making a graph of its daily activity.

C. Find four different sized snapshots. Determine what dimensions they would be if they were enlarged to three times their size. What would their dimensions be if the new size were half the original size?

D. Find out what "continued fractions" are and how they can be computed from ordinary fractions. (See *Twenty-seventh Yearbook*, pp. 247–249.)



For some minutes Alice stood without speaking, looking out in all directions over the country—and a most curious country it was. There were a number of tiny little brooks running straight across it from side to side, and the ground between was divided up into squares by a number of little green hedges, that reached from brook to brook.

"I declare, it's marked out just like a large chessboard!" Alice said at last.

9

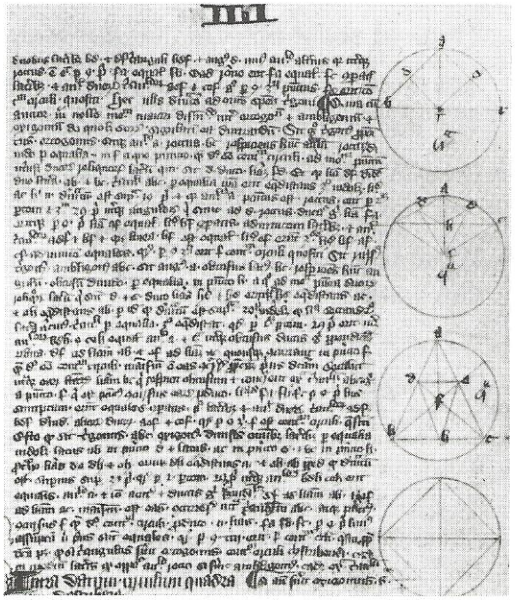
Geometry Unit III
Parallel Lines, Pythagorean Theorem

A. NONMETRIC GEOMETRY

9-1 PARALLEL AND SKEW LINES

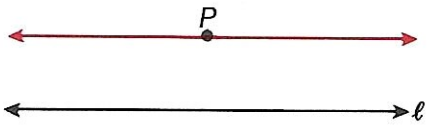
About 300 B.C., the great Greek mathematician Euclid wrote *The Elements*. In this book, he showed that all the known geometry of his time depended on a few simple geometric statements. These statements are called **postulates**, and they are mostly very simple. For example, the first postulate tells us that there is only one line segment between any two points.

Only one line segment from *A* to *B*



Part of a page from the *Fifteen Books of Euclid's Elements.* This manuscript was written in Latin by a 14th-century English scribe.

Euclid's most famous postulate is his fifth postulate, which tells us:



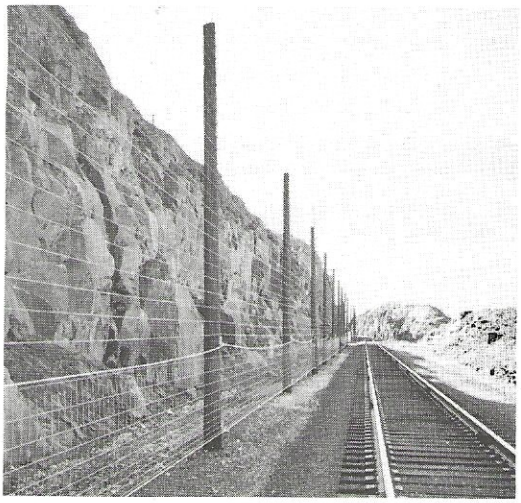
> In a plane, there is only one line through the point P that does not meet the given line ℓ.

Two lines in a plane that do not intersect are called **parallel** lines. We show that two lines m and n are parallel by writing

$$m \parallel n$$

We read, "m is parallel to n."

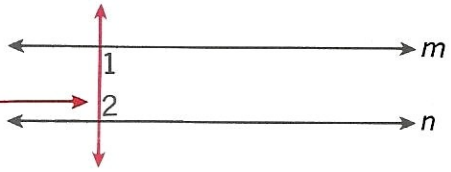
Two lines in space that are not in the same plane are called **skew** lines.



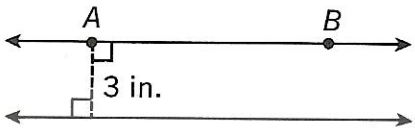
Can you see any parallel or skew lines in this picture?

Discussion Exercises

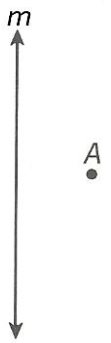
1. Think of some objects that suggest parallel lines.
2. Think of some objects that suggest skew lines, as, for example, one highway crossing above another highway.
3. If lines o and p are not parallel and are in the same plane, what can you say about them?
4. If two lines o and p are not parallel and yet do not intersect, what can you say about them?
5. If $m \parallel n$ and $\angle 1$ is a right angle, what can you say about $\angle 2$?



6. You can use a segment that is perpendicular to two parallels to measure the distance between them. If you find that the lines are 3 in. apart at point A, what do you think you will find at point B?



7. Here is a line m and a point A not on m.



[a] How many lines are there through A parallel to m?

[b] Is there more than one line in space that passes through A and does not intersect m?

★**8.** Consider two lines in space by thinking about the paths of two jet airplanes.

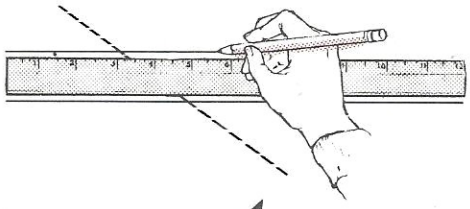
[a] If these paths are everywhere the same distance apart, what can you say about them?

[b] If the paths are skew, will there be two points, one on each line, that are closer to each other than any other such pair of points?

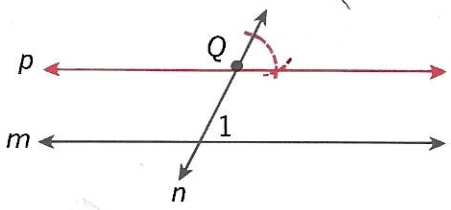
[c] What might be meant by the "distance" between two skew lines?

Exercises

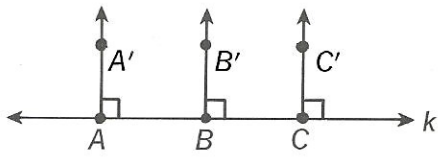
1. Draw a line on each side of your ruler. Then draw a third line (like the dotted line in the figure) that intersects the two lines. Can you find four different congruent angles? Label them.



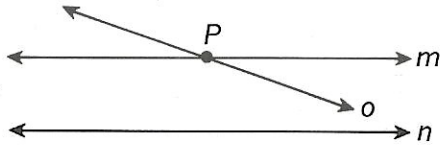
2. Draw a line m. Draw a line n crossing m at $\angle 1$. Choose a point Q on n. Copy $\angle 1$ as indicated in the figure and draw p. How do you think lines m and p are related?



3. Draw a line k. Mark three points, A, B, and C, on it. With your ruler and compass, construct three rays, all perpendicular to k as in the figure. On these rays, mark points A', B', and C' so that $\overline{AA'} \cong \overline{BB'} \cong \overline{CC'}$.



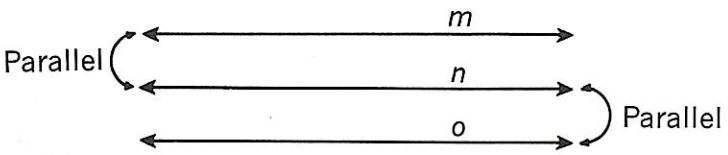
4. Lines m and n are parallel. Line o intersects m in a point P.



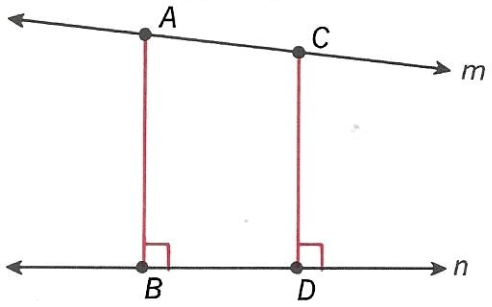
[a] What can you say about lines o and n if the lines are in the same plane?

[b] Complete the statement for three lines in the same plane: If a line intersects one of two parallel lines, then it __?__.

5. Suppose $m \parallel n$ and $n \parallel o$. What can you say about lines m and o?



★**6.** Think about two lines m and n in space as indicated in the figure. The indicated angles at B and D are right angles.



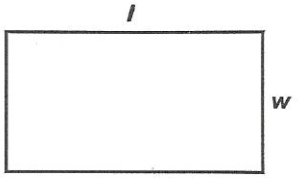
[a] If $\overline{AB} > \overline{CD}$ and m and n are in the same plane, what can you say about m and n?

[b] If $\overline{AB} > \overline{CD}$ and the lines m and n do not intersect, what can you say about m and n?

[c] Is it possible that m and n are skew lines and $\overline{AB} \cong \overline{CD}$?

[d] Is it possible to have $m \parallel n$ and have $\overline{AB}$ not congruent to $\overline{CD}$?

Think



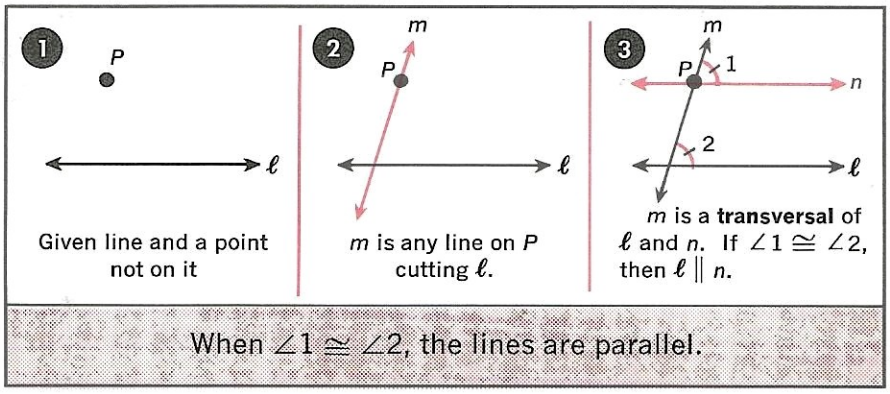
In the table, ***l*** and ***w*** represent the length and width of a rectangle. Find the length and width for each row.

Area = Half the perimeter

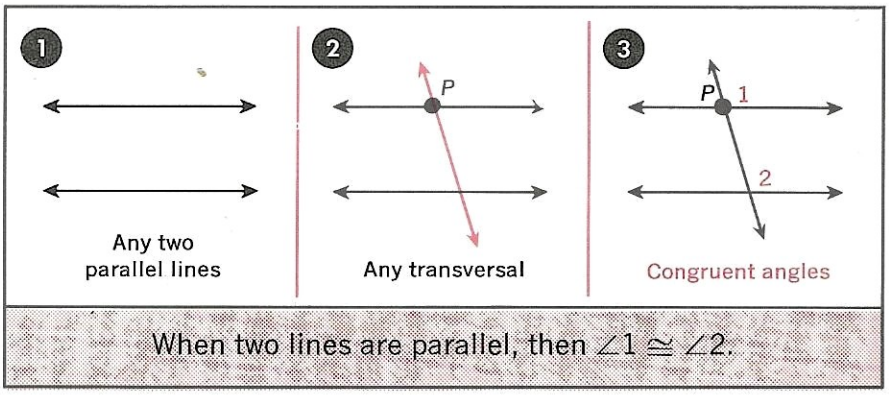
40	*l*	× = +	*w*	14
45	*l*	× = +	*w*	14
48	*l*	× = +	*w*	14
49	*l*	× = +	*w*	14

9-2 TRANSVERSALS AND PARALLEL LINES

In the last lesson, we discussed how to construct parallel lines.



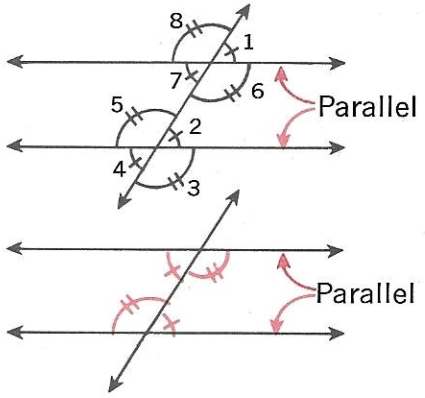
It is also interesting to note that if any two parallel lines are cut by a third line, then $\angle 1 \cong \angle 2$ as in the figure below.



Discussion Exercises

1. Angles 1 and 2 in the figure are called **corresponding angles**. Angles 3 and 6 are also corresponding angles. Find two other pairs of corresponding angles.

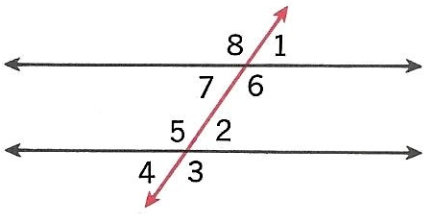
2. The four angles marked in color are called **interior angles**. Explain where you think the **exterior angles** are for this figure.



3. Angles 2 and 7 are called **alternate interior angles.**

[a] Find another pair of alternate interior angles.

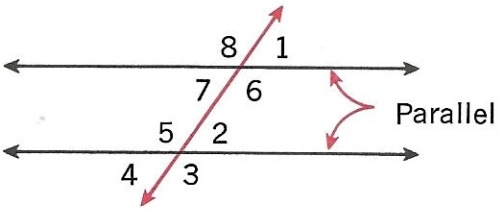
[b] Find two pairs of alternate exterior angles.



Exercises

1. [a] Name four angles all congruent to each other.

[b] Name four other angles all congruent to each other.



2. Two lines *m* and *n* are parallel. If a third line o cuts *m* at right angles, what can you say about o and *n*? Draw and label a figure to illustrate this.

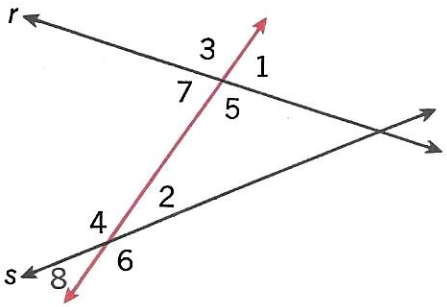
3. In the figure, there are two intersecting lines *r* and *s* cut by a transversal.

[a] Which angle is larger, $\angle 1$ or $\angle 2$?

[b] How do $\angle 7$ and $\angle 8$ compare?

[c] Compare $\angle 3$ and $\angle 4$.

[d] How do $\angle 5$ and $\angle 6$ compare?



4. Here are four ways that three lines can be located in a plane. Show each of these ways with a figure.

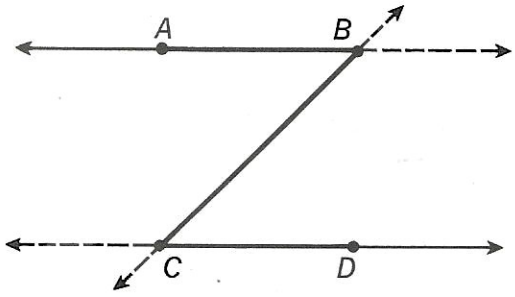
[a] No points of intersection

[b] Exactly one point of intersection

[c] Exactly two points of intersection

[d] Exactly three points of intersection

5. If $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, name two congruent angles. (The **Z**-shaped figure indicated by path *ABCD* is an important one to remember.)



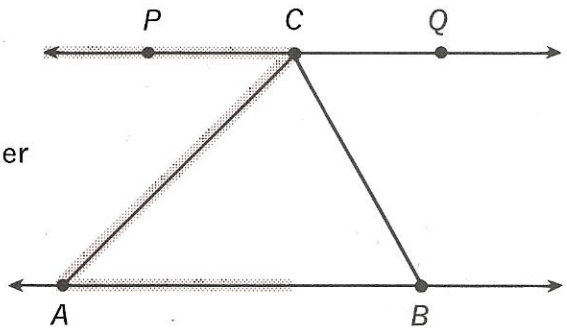
6. Draw and label a "backward" **Z**. Find a pair of congruent angles for this figure.

7. In the figure, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{AB}$.

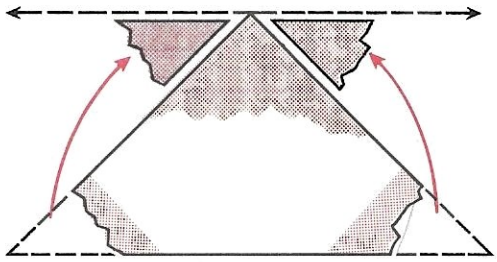
[**a**] Find a pair of congruent angles. (The **Z** should help.)

[**b**] Think of a backward **Z**, and find another pair of congruent angles.

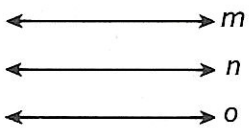
[**c**] Using the results from Parts **a** and **b**, what can you say about the sum of the angles of a triangle?



8. Draw a triangle and cut it out from your paper. Tear off two corners and place them as illustrated in the figure. What do you observe?

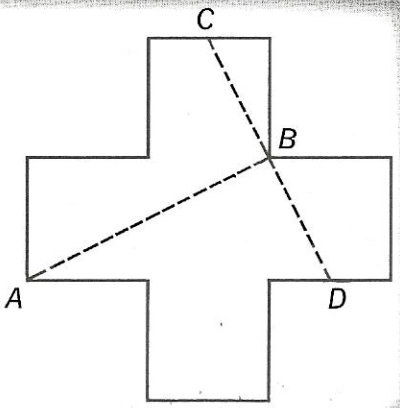


★**9.** If $n \parallel m$ and $o \parallel m$, can n and o intersect? Prove that n and o do not intersect. (**Hint**: Complete the sentence with a phrase about parallel lines. If we assume that n and o do intersect, then through one point there would be _?_.)

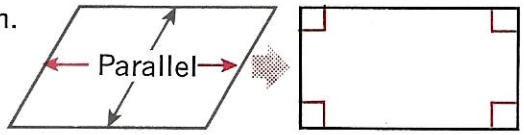


Think

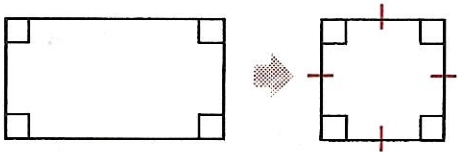
Trace the cross and the dotted lines. Cut out your tracing and then cut along the dotted lines. Now fit four pieces together to make a square.



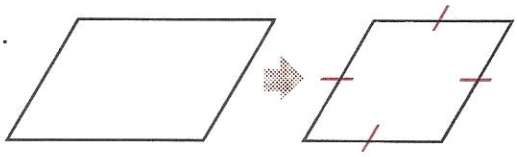
3. A **rectangle** is a special kind of parallelogram. What special feature does a rectangle have that sets it apart from other parallelograms?



4. A **square** is a special kind of rectangle. What special feature does a square have that sets it apart from the other rectangles?



5. A **rhombus** is a special kind of parallelogram. What special feature does a rhombus have that sets it apart from other parallelograms?



Exercises

1. Of the five special kinds of quadrilaterals illustrated, tell which ones have

[a] four sides

[b] at least one pair of parallel sides

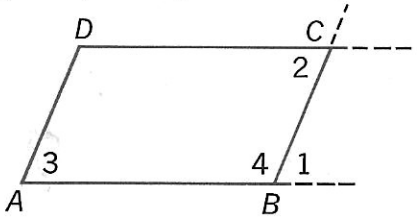
[c] two pairs of parallel sides

[d] all right angles

[e] all sides congruent

[f] only one pair of parallel sides

2. Figure *ABCD* is a parallelogram.



[a] Are $\angle 1$ and $\angle 2$ congruent? Why?

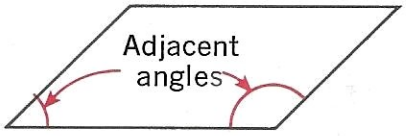
[b] Are $\angle 1$ and $\angle 3$ congruent? Why?

[c] Since $\angle 1 \cong \angle 2$ and $\angle 1 \cong \angle 3$, we know that $\angle 2$ _?_ $\angle 3$.

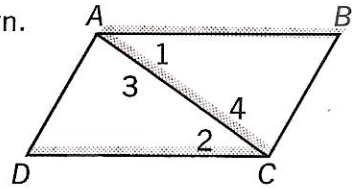
[d] The opposite angles of a parallelogram are _?_.

[e] Angle 1 and $\angle 4$ are supplementary. We also know that $\angle 1 \cong \angle 3$. Thus we know that $\angle 3$ and $\angle 4$ are _?_.

[f] Use the appropriate word (**complementary**, **supplementary**, or **congruent**) to complete the sentence: Two adjacent angles of a parallelogram are _?_



3. Figure *ABCD* is a parallelogram with one **diagonal** drawn.

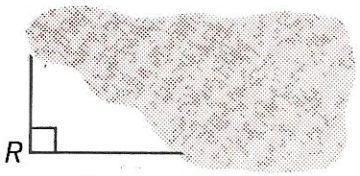


[a] What can you say about $\angle 1$ and $\angle 2$?

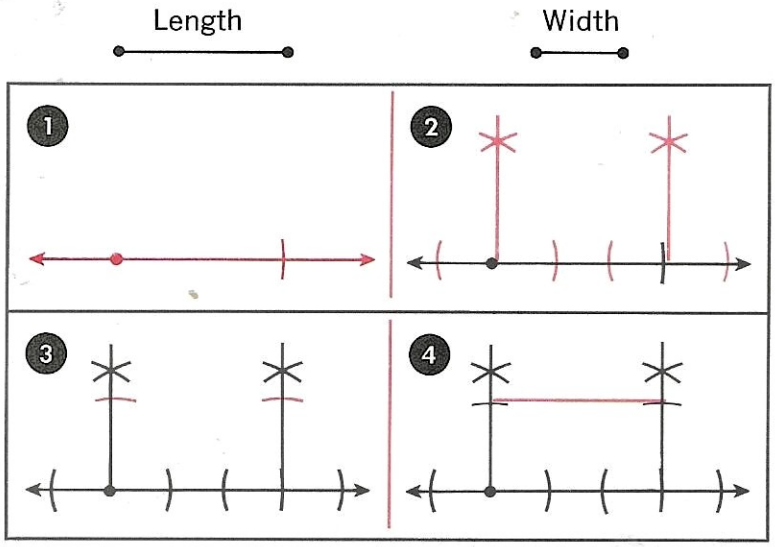
[b] Is $\angle 3 \cong \angle 4$?

[c] If $\triangle DCA \cong \triangle BAC$, what can you say about the opposite sides of a parallelogram?

4. All but one corner of a parallelogram *RSTU* is covered. What can you say about the parallelogram if you know that $\angle R$ is a right angle?



5. Steps for constructing a rectangle with sides as indicated are given below.

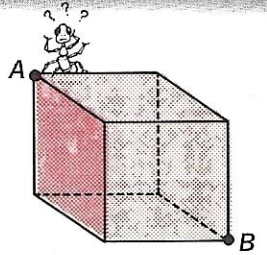


Using this method, construct a rectangle for the following sides:

length ●——————● width ●——————●

Think

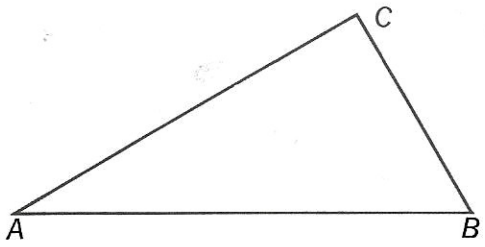
Suppose a bug is at point *A*. What is the shortest path he can crawl in order to get to point *B*? Draw this path.



B. METRIC GEOMETRY

9-4 ANGLE MEASURE—SUMS

We have discovered that the three angles of a triangle, "taken together," form a straight angle. Using measurement, we can now show this same idea. If we draw any triangle and carefully measure the three angles, we should find that the sum of the measures is very close to 180°.



$m\angle A + m\angle B + m\angle C = 180°$

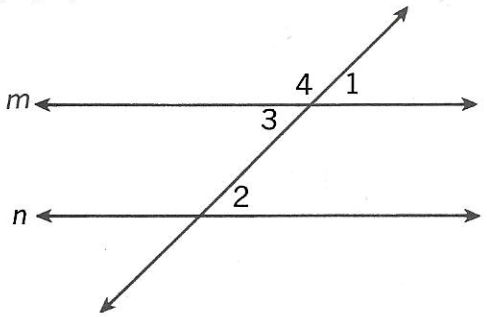
Exercises

1. Lines m and n are parallel. Measure angles 1, 2, 3, and 4.

[a] Which angles are a pair of alternate interior angles? Do they have the same measure?

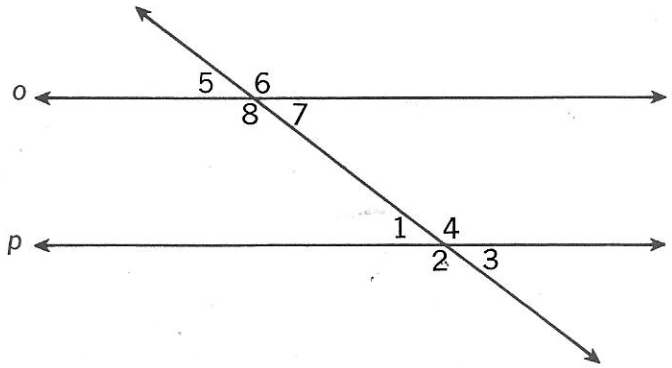
[b] Which angles are a pair of corresponding angles? Do they have the same measure?

[c] Name a pair of angles that is supplementary. What is the sum of the angle measures?

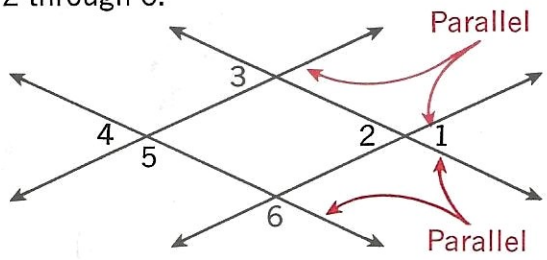


2. Lines o and p are parallel. If $\angle 1$ has a measure of 37°, give the measure of

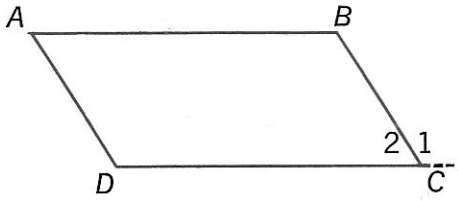
[a] $\angle 3$ [b] $\angle 7$ [c] $\angle 6$ [d] $\angle 8$ [e] $\angle 4$



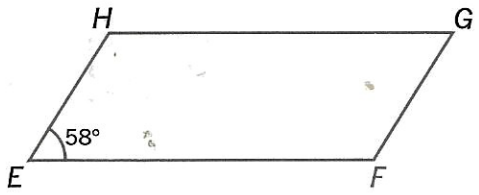
3. The lines are parallel as indicated in the figure. Angle 1 has a measure of 50°. Give the measure of angles 2 through 6.



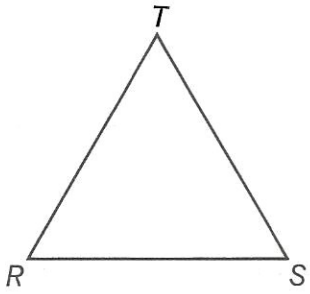
4. Figure *ABCD* is a parallelogram.
[a] If $m\angle 1 = 120°$, what is $m\angle 2$?
[b] If $m\angle 1 = 125°$, what is $m\angle D$?
[c] If $m\angle 1 = 115°$, what is $m\angle B$?
[d] If $m\angle 1 = 130°$, what is $m\angle A$?



5. In parallelogram *EFGH*, $\angle E$ has a measure of 58°.
[a] Give the measure of angles *F*, *G*, and *H*.
[b] What is the sum of the measures of all four angles?

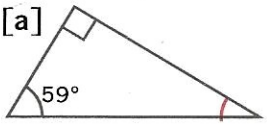


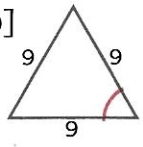
6. Triangle *RST* is equilateral (all sides congruent). What is the measure of each angle?



7. [a] What is the measure of a right angle?
[b] What is the sum of the measures of two right angles?
[c] From Part **b**, you could say that the sum of the angles of a triangle is two _?_.

8. The degree measures of some of the angles are shown on the figures. Find the measures of the angles indicated by color.

[a]


[b]


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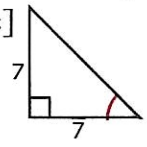

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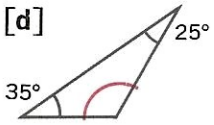

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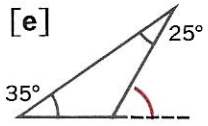

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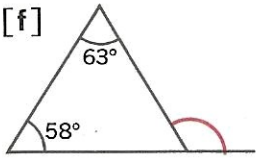

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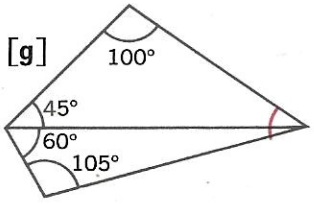

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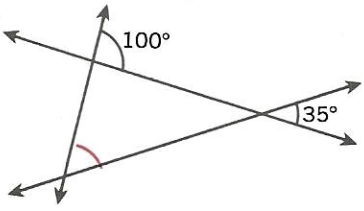
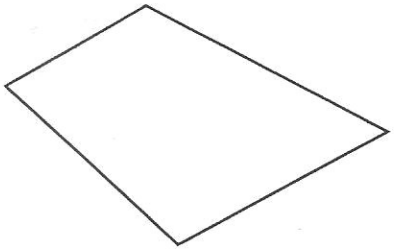
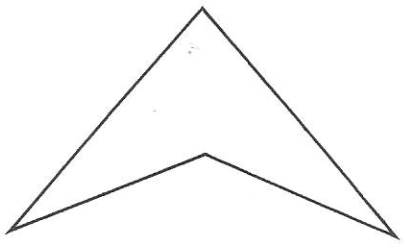


9-3 QUADRILATERALS

The pictures illustrate the general idea of **quadrilateral** figures. The following two figures are "closed" four-sided quadrilaterals.

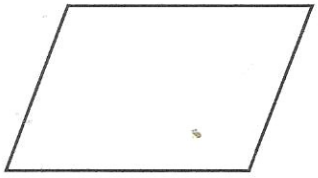


Convex

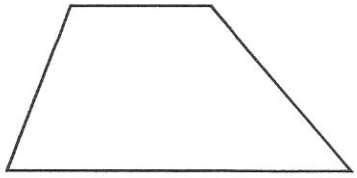


Concave

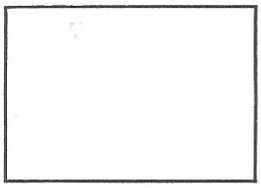
Some special kinds of quadrilaterals are illustrated in the figures below.



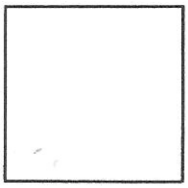
Parallelogram



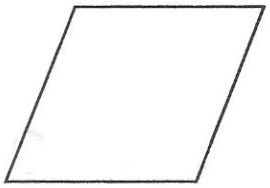
Trapezoid



Rectangle



Square



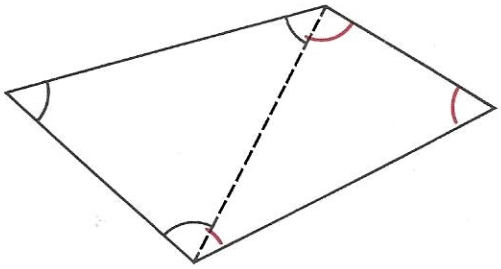
Rhombus

Discussion Exercises

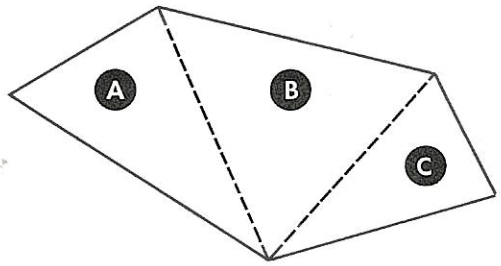
1. A **trapezoid** is usually defined as a quadrilateral with one and only one pair of parallel sides.
   [**a**] How would you define a parallelogram?
   [**b**] Is a parallelogram a trapezoid?
   [**c**] Is it possible to have a trapezoid that is also a parallelogram?

2. Describe in your own words the difference between a **convex** and a **concave** quadrilateral.

9. [a] What is the sum of the measures of the angles marked in black?
[b] What is the sum of the measures of the angles marked in color?
[c] What is the sum of all the angles?
[d] What is the sum of the angles of any quadrilateral?

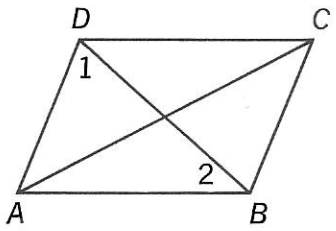


10. [a] What is the sum of the angles for Triangle A?
[b] What is the sum for each of Triangles B and C?
[c] What is the sum of all the angles?
[d] What is the sum of the angles of a pentagon?



★ **11.** What is the sum of the angles of a hexagon?

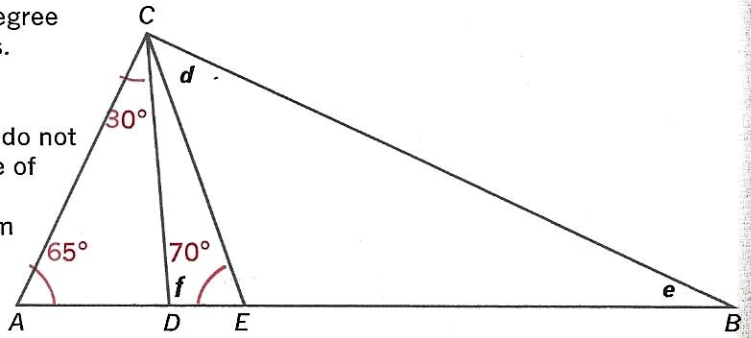
★ **12.** If $m\angle 1 = 70°$ and $m\angle 2 = 42°$, what are the measures of the angles of the parallelogram?



Think

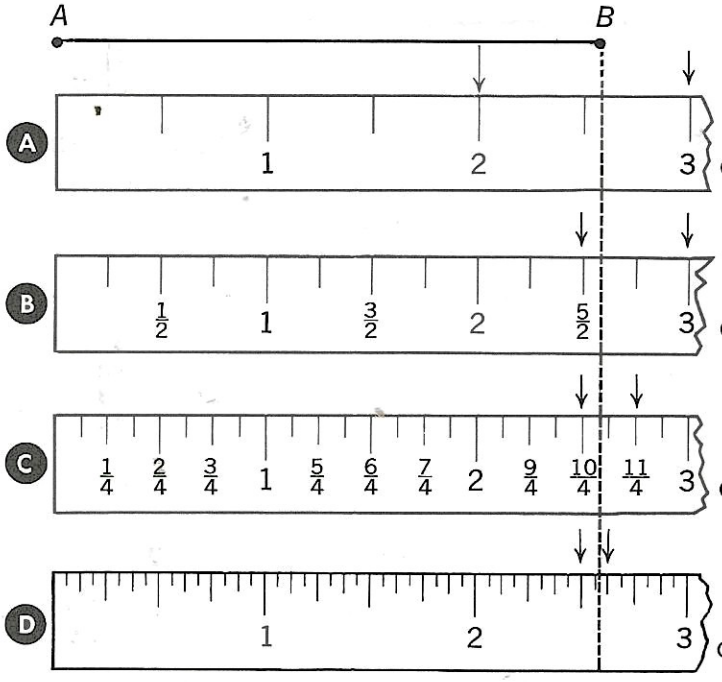
Triangle *ABC* is a right triangle with the right angle at C. The small letters represent the degree measures of the angles.

1. Find ***d***, ***e***, and ***f***.
2. Suppose that you do not know the measure of $\angle ACB$. Can you determine the sum $d + e + f$?



9-5 RATIONAL NUMBERS IN MEASUREMENT

Whether we are measuring a distance, an angle, a weight, or a period of time, we use the same basic idea as we perform the measurement. We choose a unit and compare it with the thing being measured. When we use a ruler, our unit is usually 1 inch, although we may use $\frac{1}{2}$, $\frac{1}{4}$, or $\frac{1}{8}$ inch, to express the measure. For example, let us measure the segment AB, using the inch as the unit.



To the nearest inch, the length of this segment is 3 in.

To the nearest $\frac{1}{2}$ in., the length of this segment is $\frac{5}{2}$ in.

To the nearest $\frac{1}{4}$ in., the length of this segment is $\frac{10}{4}$ in.

To the nearest $\frac{1}{8}$ in., the length of this segment is $\frac{21}{8}$ in.

Discussion Exercises

1. Give the distance between the arrows placed above
[a] Ruler A [b] Ruler B [c] Ruler C [d] Ruler D

2. How can you tell, using Ruler A, that the length of $\overline{AB}$ is nearer 3 than 2 in?

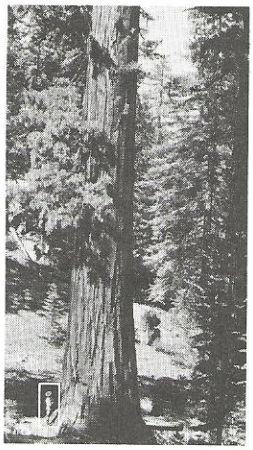
3. How can you tell, using Ruler B, that the length of $\overline{AB}$ is nearer $\frac{5}{2}$ than $\frac{6}{2}$ in?

4. A forest ranger tells you that a giant sequoia is about 1800 yr old.

[a] What unit of time do you think he is using?

[b] Would you feel that he had given you inaccurate information if you learned that the tree was between 1830 and 1840 yr old?

[c] Would you feel that he had given you inaccurate information if you learned that the tree was actually between 900 and 950 yr old?



A giant sequoia

5. A friend uses a micrometer to measure the thickness of a sheet of paper and tells you it is about $\frac{2}{1000}$ in. thick.

[a] Would you feel that he had given you inaccurate information if you found that a stack of 450 sheets was about 1 in. thick?

[b] Would you feel that he had given you inaccurate information if you found that 200 sheets of this paper made a stack 1 in. thick?

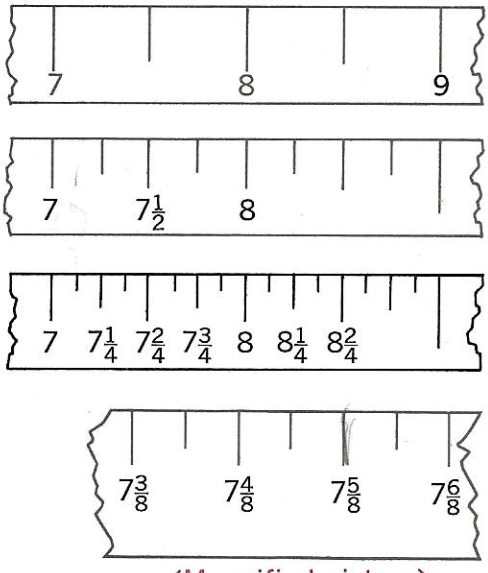
Exercises

1. A line segment measures 8 in. to the nearest inch. Between what two limits does its length lie?

2. A line segment measures $7\frac{1}{2}$ in. to the nearest $\frac{1}{2}$ in. Between what two limits does its length lie?

3. A line segment measures $7\frac{3}{4}$ in. to the nearest $\frac{1}{4}$ in. Between what two limits does its length lie?

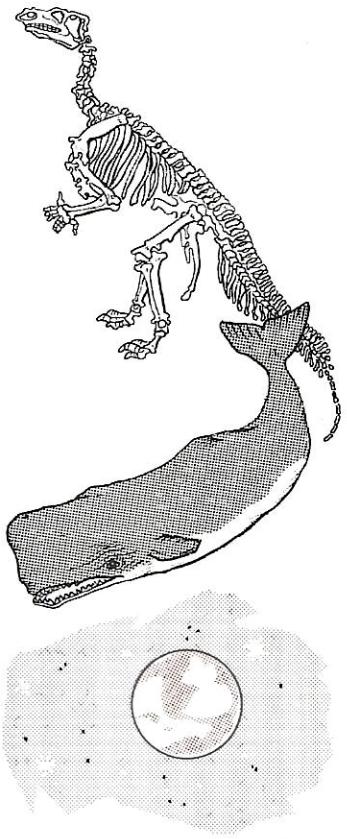
4. A line segment measures $7\frac{5}{8}$ in. to the nearest $\frac{1}{8}$ in. Between what two limits does its length lie?



(Magnified picture)

5. The measurement of Exercise 2 could be described as $7\frac{1}{2} \pm \frac{1}{4}$ in. In the same way, describe the measurement of

[a] Exercise 3 [b] Exercise 4



6. The age of a skeleton is estimated by a scientist to be 500,000 $\pm$ 50,000 yr. Between what limits does the scientist believe the age of the skeleton lies?

7. Is the scientist of Exercise 6 estimating the age of the skeleton to the nearest 10,000 yr; 50,000 yr; or 100,000 yr?

8. The weight of a whale is given as 13 tons to the nearest ton. If this weight is accurately reported, between what two weights in pounds does the whale's weight lie?

9. The lifetime of a subatomic particle, the muon, is given by a scientist as 2 millionths of a second. Would you expect most muons to have a life of less than 1 millionth of a second? More than 3 millionths of a second?

10. A planet's year is the time required to revolve about the sun. An earth year is a little less than $365\frac{1}{4}$ earth days. A table gives the length of a year on Mars as 687 earth days. Accepting this measurement, is it possible that a year on Mars might be

[**a**] as long as 687 days 10 hr?

[**b**] as short as 686 days 10 hr?

11. When a measurement is reported as 14 to the nearest unit, the actual measure is expected to fall between $13\frac{1}{2}$ and $14\frac{1}{2}$ units. The **greatest possible error** resulting from using 14 as the actual measure is $\frac{1}{2}$ unit. Give the greatest possible error associated with each of the following measurements.

[**a**] 15 in. to the nearest inch

[**b**] 20 mi to the nearest mile

[**c**] 4 light yr to the nearest light year

[**d**] 13 oz to the nearest ounce

[**e**] 51 sec to the nearest second

[**f**] $7\frac{3}{4}$ in. to the nearest $\frac{1}{4}$ in.

[**g**] 8 in. to the nearest $\frac{1}{8}$ in.

[**h**] 8 in. to the nearest $\frac{1}{16}$ in.

[**i**] 8 in. to the nearest $\frac{1}{32}$ in.

[**j**] $21\frac{1}{2}$ sec to the nearest $\frac{1}{10}$ sec

12. Write the measures of Exercise 11 in the "$\pm$" notation. For example, 11**a** would be 15 $\pm \frac{1}{2}$ in. and **f** would be $7\frac{3}{4} \pm \frac{1}{8}$ in.

13. Astronomers give the average distance from Jupiter to the sun as 483 million mi. Accepting this measurement, could this average distance be as much as 483,700,000 mi? Why or why not?

14. An astronomer says it is about 12 light yr from earth to a certain star. Taking a light year as 6 trillion mi, how much might the actual distance differ from the astronomer's figure?

★**15.** Towns *A*, *B*, *C* are on the same highway. The distance from *A* to *B* is 32 mi to the nearest mile. The distance from *B* to *C* is 24 mi to the nearest mile. The distance from *A* to *C* is 55 mi to the nearest mile. Explain how this could be the case.

★**16.** When we compare the **precision** of two measurements, we compare their greatest possible errors. The measurement with the smaller greatest possible error is said to be **more precise** than the other. A measurement given as $8 \pm \frac{1}{16}$ in. is more precise than one given as $17 \pm \frac{1}{4}$ in. Pick the more precise measurement from each of the following pairs.

[a] $7\frac{1}{2} \pm \frac{1}{8}$ in., $10 \pm \frac{1}{4}$ in.
[b] $12 \pm \frac{1}{100}$ ft, $14 \pm \frac{1}{8}$ in.
[c] $7\frac{1}{2} \pm \frac{1}{4}$ oz, $4 \pm \frac{1}{100}$ lb
[d] $18 \pm \frac{1}{2}$ sec, $7 \pm \frac{1}{100}$ min

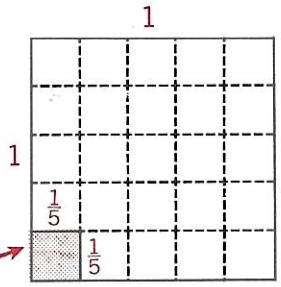
★**17.** We find the **relative error** of a measurement by dividing the greatest possible error by the measurement. For example, if a measurement is reported as $20 \pm \frac{1}{2}$ in., the relative error $= \frac{1}{2} \div 20 = \frac{1}{40}$. We say that the error might be as much as 1 part in 40. When we compare two measurements, the one with the smaller relative error is said to be **more accurate**.

Example: Compare the measurements 20 ft ± 3 in. and $4 \pm \frac{1}{8}$ in. for accuracy: $3 \div 240 = \frac{1}{80}$; $\frac{1}{8} \div 4 = \frac{1}{32}$.
Since $\frac{1}{80} < \frac{1}{32}$, the first measurement is more accurate than the second. Notice that the second is more precise than the first.
From each following pair of measurements, select the more accurate.

[a] $8 \pm \frac{1}{2}$ in., $10 \pm \frac{1}{2}$ in.
[b] 500 ± 1 ft, $14 \pm \frac{1}{32}$ in.
[c] 8000 ± 10 mi, $40 \pm \frac{1}{16}$ in.
[d] 255,000 ± 1000 yr, $20 \pm \frac{1}{10}$ sec

9-6 AREA

The figure at the right illustrates how the unit square can be divided into twenty-fifths. Just as for length, dividing the unit into smaller pieces allows us to make more accurate measurements for area. For example, to find the area of this region, we count small square regions that are $\frac{1}{25}$ of the unit region.



There are 25 of these smaller square regions in the unit region.

1 sq unit → 25 small squares

→ 15 small squares

→ 10 small squares

→ 6 small squares

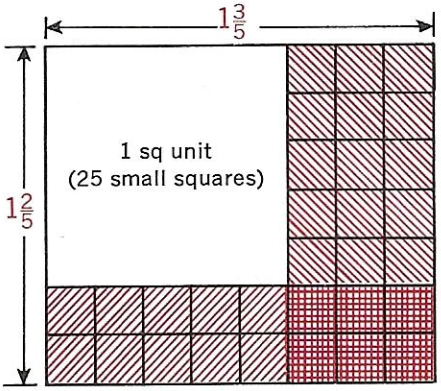
Total area → 56 small squares

Each small square is $\frac{1}{25}$ of a square unit.

$56 \cdot \frac{1}{25} = 2\frac{6}{25}$

Area $= 2\frac{6}{25}$ sq units

Notice that $1\frac{3}{5} \cdot 1\frac{2}{5} = \frac{8}{5} \cdot \frac{7}{5} = \frac{56}{25} = 2\frac{6}{25}$



You will recall that the formula for the area of a rectangle, when l and w are whole numbers, is $A = l \cdot w$. The preceding example should help you see that this same formula holds when l and w are rational numbers.

Discussion Exercises

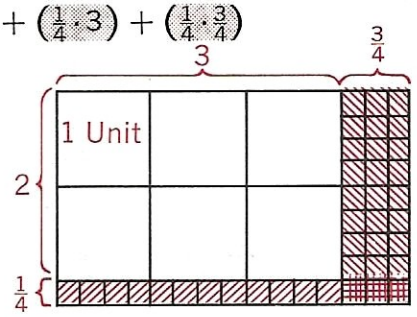
1. [a] Tell what region each screened product in the equation represents.

$$2\tfrac{1}{4} \cdot 3\tfrac{3}{4} = (2 + \tfrac{1}{4}) \cdot (3 + \tfrac{3}{4}) = (2 \cdot 3) + (2 \cdot \tfrac{3}{4}) + (\tfrac{1}{4} \cdot 3) + (\tfrac{1}{4} \cdot \tfrac{3}{4})$$

[b] What part of a unit region is each small square region in the figure?

[c] Thinking about the small $\frac{1}{4}$ by $\frac{1}{4}$ region as the unit, what is the total number of small squares for the rectangle?

[d] What is the area of the rectangle, using the unit indicated?

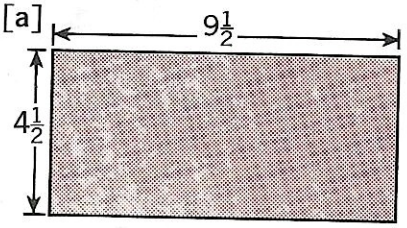


2. Does the formula $A = \frac{1}{2} \cdot b \cdot h$ hold for the area of a triangle when b and h are rational numbers? Explain your answer.

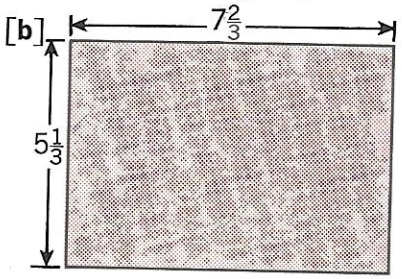
Exercises

1. Find the area of each region, using the dimensions given.

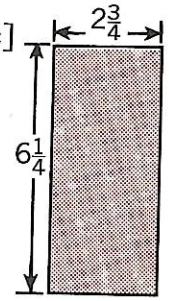
[a]



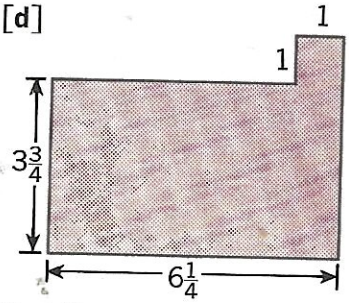
[b]



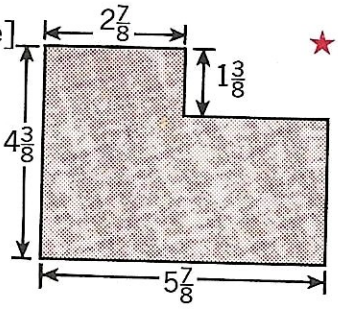
[c]



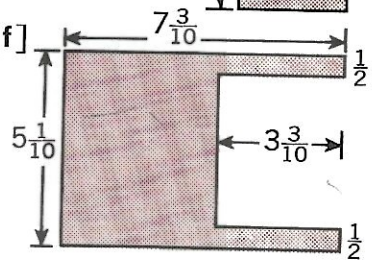
[d]



[e]

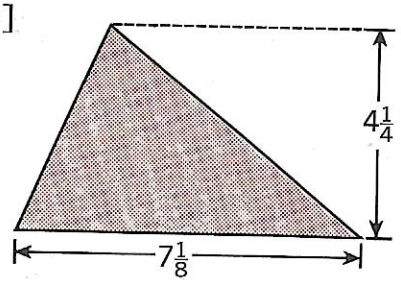


★[f]

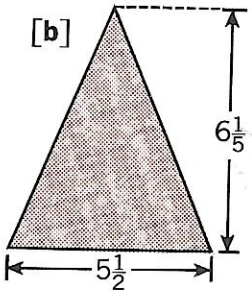


2. Find the area of each region, using the dimensions given.

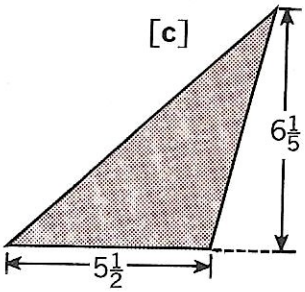
[a]



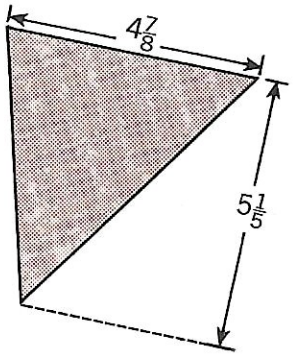
[b]



[c]



[d]



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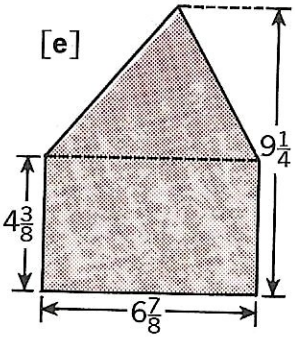
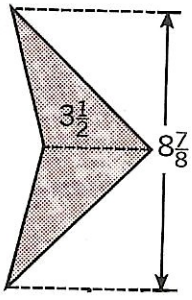


[f]

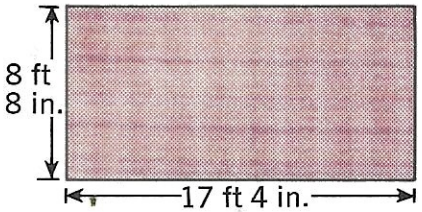


3. Find the floor area in square feet of a room 11 ft 3 in. by 15 ft 6 in.

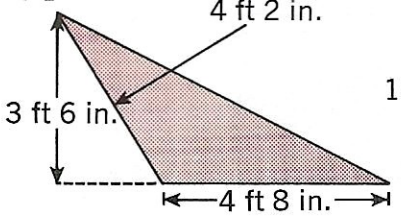
4. A room 13 ft by 17 ft has a floor area of how many square yards?

5. Find the area in square feet of each figure.

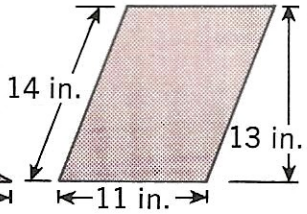
[a]



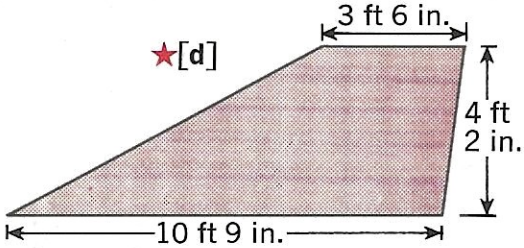
[b]



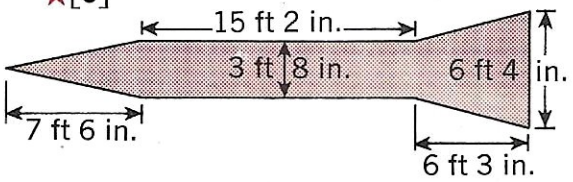
[c]



★[d]



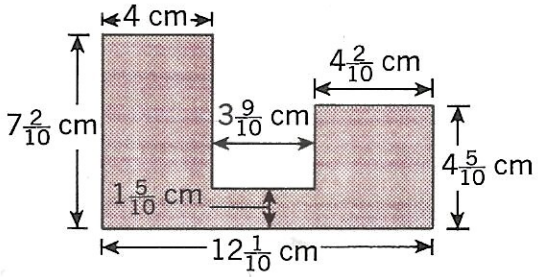
★[e]



★**6.** Find the area in

[a] square centimeters

[b] square millimeters



★**7.** The measured length and width of a rectangle are given as 8 in. and 5 in. to the nearest inch.

[a] Show that the area might be a little more than 46 sq in.

[b] Show that the area might be a little less than 34 sq in.

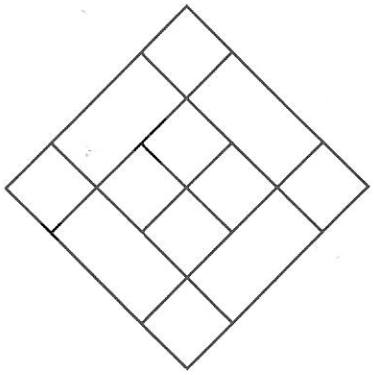
[c] If we describe the area of the rectangle as $40 \pm n$ sq in., what is a reasonable number for n?

★**8.** A rectangle is $1\frac{3}{8}$ by $2\frac{3}{4}$ in. Draw a picture to show the nine rectangles (and find their areas) suggested by writing the following:

$$1\tfrac{3}{8} = 1 + \tfrac{1}{4} + \tfrac{1}{8} \quad \text{and} \quad 2\tfrac{3}{4} = 2 + \tfrac{1}{2} + \tfrac{1}{4}$$

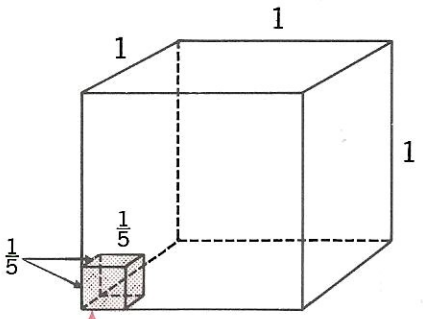
Think

How many different squares can you find in the figure?



9-7 VOLUME

The figure at the right illustrates how the unit cube can be divided into one hundred twenty-fifths. Just as for area, dividing the unit into smaller pieces allows us to make more accurate measurements for volume.

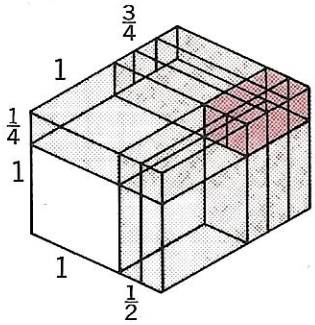


There are 125 of these smaller cubic regions in the unit region.

The figure at the right illustrates the volume of a box with dimensions of $1\frac{1}{2}$, $1\frac{1}{4}$, and $1\frac{3}{4}$. As with area, the familiar formula for volume

$$V = l \cdot w \cdot h$$

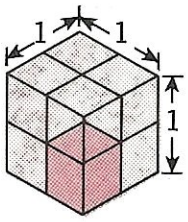
holds when l, w, and h are rational numbers.



Discussion Exercises

1. Find the volume for the box in the preceding figure.

2. The figure shows a unit cube with smaller cubes half as long on a side.



[a] What is the volume measure of each small cube?

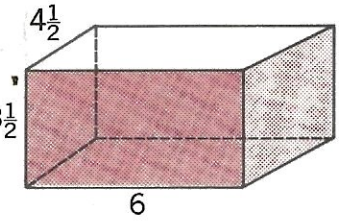
[b] What would be the volume measure of small cubes $\frac{1}{3}$ as long as the unit on a side?

3. The odder the shape, the harder the problem of finding the volume. For practical uses, do you need a formula for the volume of the coffee pot? What would be the most convenient unit for giving the volume of a coffee pot?

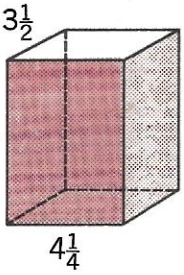
Exercises

1. Find the volume of each solid, using the dimensions given.

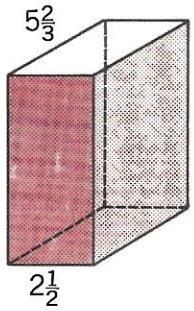
[a] $4\frac{1}{2}$, $3\frac{1}{2}$, 6



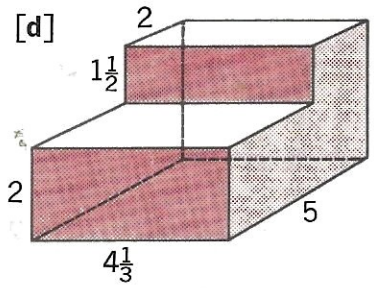
[b] $3\frac{1}{2}$, 6, $4\frac{1}{4}$



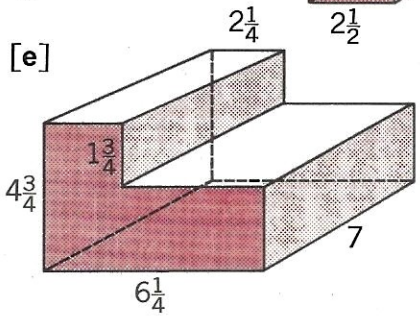
[c] $5\frac{2}{3}$, $7\frac{1}{4}$, $2\frac{1}{2}$



[d] 2, $1\frac{1}{2}$, 2, 5, $4\frac{1}{3}$

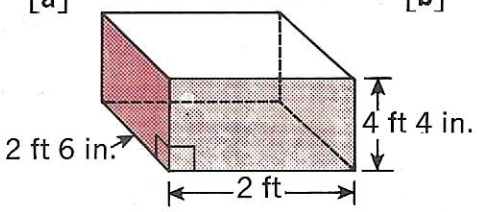


[e] $2\frac{1}{4}$, $1\frac{3}{4}$, $4\frac{3}{4}$, 7, $6\frac{1}{4}$

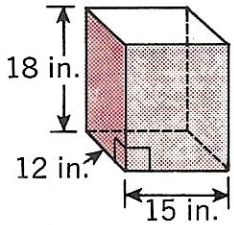


2. Find the volume of each solid in cubic feet.

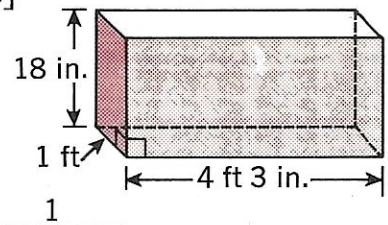
[a] 2 ft 6 in., 4 ft 4 in., 2 ft



[b] 18 in., 12 in., 15 in.



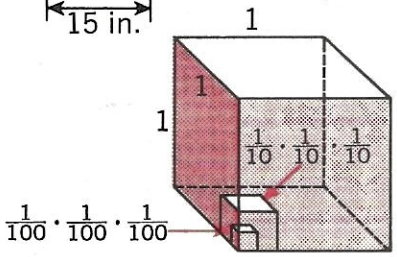
[c] 18 in., 1 ft, 4 ft 3 in.



3. Think of the large cube as the unit.

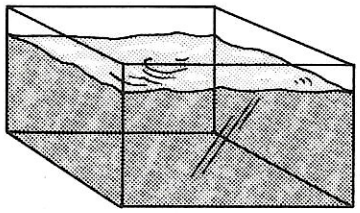
[a] How many cubes $\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10}$ are in the large cube?

[b] How many cubes $\frac{1}{100} \cdot \frac{1}{100} \cdot \frac{1}{100}$ are in the large cube?



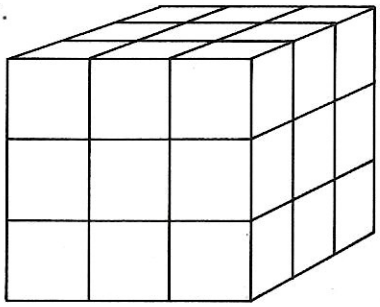
(Figure not drawn to scale)

★**4.** A rectangular tank is $\frac{3}{4}$ full of water. The dimensions of the tank are 2 ft 3 in. by 2 ft 6 in. by 1 ft 4 in. How many gallons of water are in the tank? (1 gal = 231 cu in.)



Think

Twenty-seven blocks were stacked as in the figure. Then each face of the large cube was painted.



1. Tell how many blocks have paint on

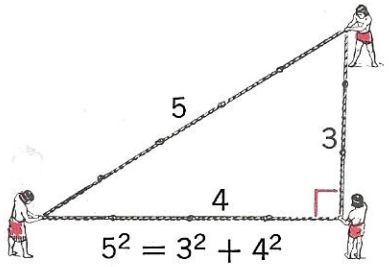
[a] one face **[c]** three faces
[b] two faces **[d]** four faces

2. How many blocks have no paint on them?

3. Answer Exercises 1 and 2 as if the painter had forgotten to lift the block and paint the bottom.

9-8 THE THEOREM OF PYTHAGORAS

There is a legend that the ancient Egyptians, surveying the land after floods, formed right angles by stretching ropes to form a 3 by 4 by 5 right triangle.



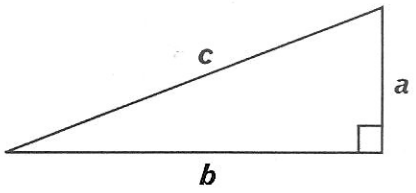
In the sixth century B.C., the Greek philosopher Pythagoras (who lived in the Greek colony of Crotona in Italy) is believed to have proved the general result for a right triangle.

Pythagoras, from a fresco by Raphael

The Pythagorean Theorem

In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

$$c^2 = a^2 + b^2$$



a = length of a leg
b = length of a leg
c = length of hypotenuse

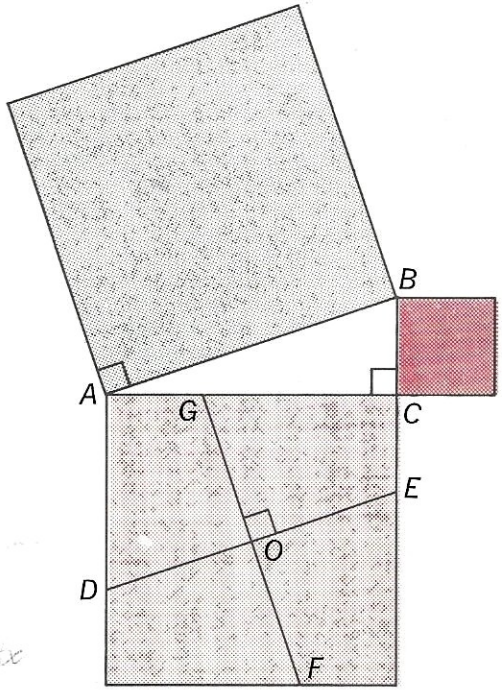
In the sequence below, you are given an opportunity to demonstrate the truth of the Pythagorean Theorem. By following the directions, you will be able to give a convincing visual argument that the square on the hypotenuse is equal to the sum of the other two squares.

1. Construct a right triangle with squares on the sides as in the figure.

2. Find point O, the center of the middle-sized square.

3. Draw $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$; then draw $\overleftrightarrow{GF} \perp \overleftrightarrow{DE}$. ($\overleftrightarrow{GF}$ contains point O.) Four regions are formed in the middle-sized square.

4. Cut out the four regions and the small square region. Fit them over the large square region.



Exercises

1. The figures are four right triangles. A square region is shaded on each side of each triangle. You can find the area of the squares on the legs simply by counting small unit squares. You can find the area of the square on the hypotenuse by thinking about the square and triangular regions suggested by the heavy colored lines. Copy and complete the table below the figures.

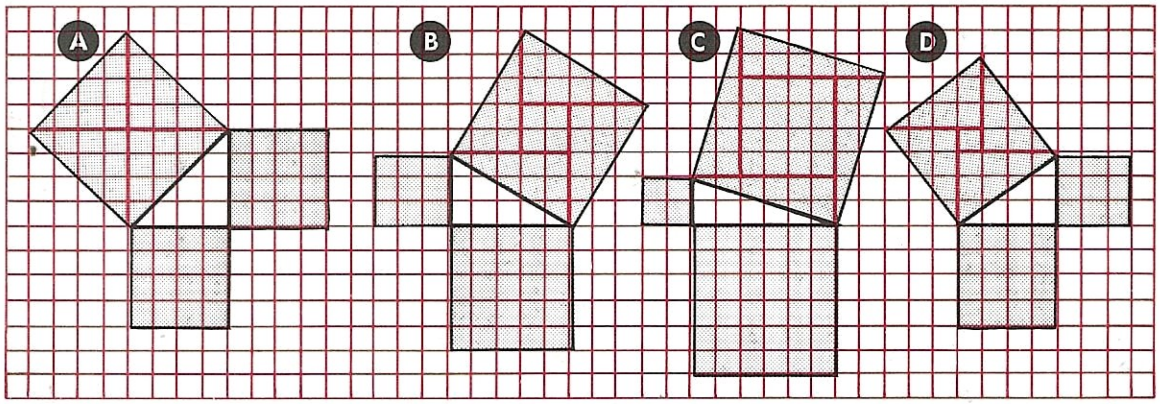
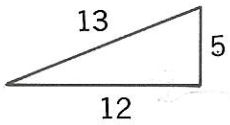


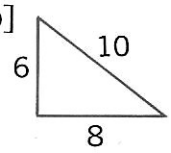
Figure	Area of the two small squares		Sum of areas of two small squares	Area of large square
A				
B				
C				
D				

2. Which triangle is not a right triangle?

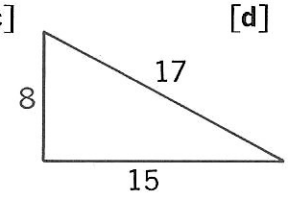
[a]



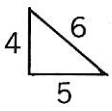
[b]



[c]

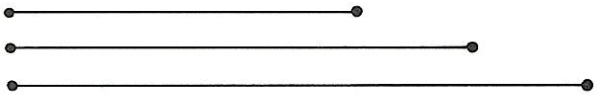


[d]



3. Construct a triangle with the sides:

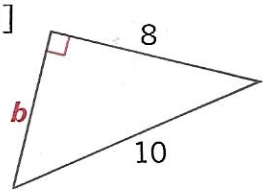
If you do the construction carefully, you should have a right triangle.



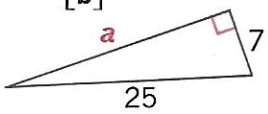
4. Construct any right triangle. Measure the sides to the nearest $\frac{1}{8}$ in. Verify the Pythagorean Theorem for your triangle.

★5. Find the length of ***a***, ***b***, or ***c*** as indicated.

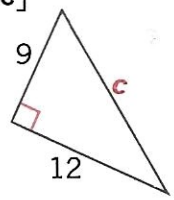
[a]



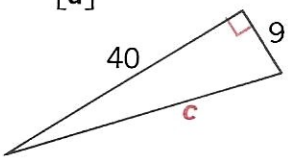
[b]



[c]

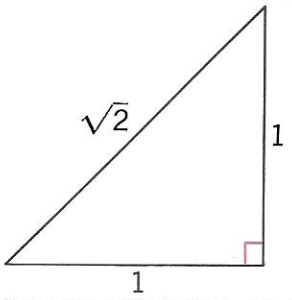


[d]



Think

The symbol $\sqrt{\ }$ is used to denote the **square root** of a number. Thus: $\sqrt{4} = 2$, $\sqrt{9} = 3$, $\sqrt{25} = 5$, $\sqrt{a^2} = a$ $(a > 0)$, and $\sqrt{2}$ is a number more than 1 but less than 2. The number $\sqrt{2}$ can be constructed geometrically as shown. Construct a segment of length $\sqrt{3}$.



CHAPTER REVIEW

Key Concepts

1. Two lines in the same plane that do not intersect are **parallel**.
2. Two lines that are not in the same plane are **skew** lines.
3. Through a point not on a line, there is just one line parallel to the given line.
4. If a transversal cuts two parallel lines, it forms pairs of congruent angles called alternate interior angles, corresponding angles, and alternate exterior angles.
5. The sum of the angles of a triangle is two right angles or 180°.
6. Opposite sides and opposite angles of a parallelogram are congruent.
7. A **quadrilateral** has four sides. A **parallelogram** is a quadrilateral with opposite sides parallel. A **rectangle** is a parallelogram with four right angles. A **square** is a rectangle with four congruent sides. A **trapezoid** is a quadrilateral with one and only one pair of parallel sides.
8. In many instances, it is important to state the limits for a given measure so that the accuracy or precision can be determined.
9. In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

Vocabulary List

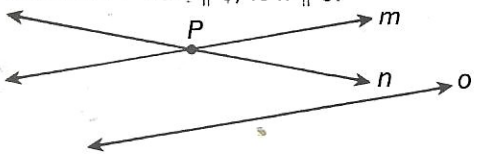
accuracy
alternate exterior angles
alternate interior angles
concave
convex
corresponding angles
greatest possible error
parallel
parallelogram
postulate
precision
quadrilateral
rectangle
relative error
rhombus
skew
square
theorem
transversal
trapezoid

Review Exercises

1. If two lines are in the same plane and are not parallel, what can you say about them?

2. If two lines are not in one plane, can they be parallel?

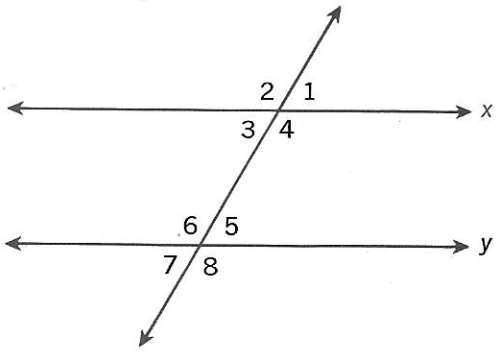
3. Lines m, n, and o are in a plane; m and n meet at P. If $m \parallel o$, is $n \parallel o$?



4. Lines r and s are parallel. Line t intersects line r. Can t and s be skew?

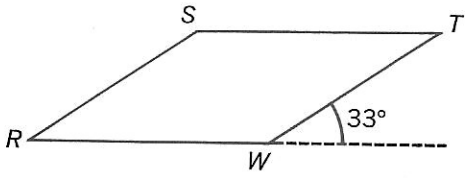
5. In the figure, lines x and y are parallel. Name all the pairs of indicated angles.

[a] Vertical
[b] Corresponding
[c] Alternate interior
[d] Alternate exterior
[e] Supplementary

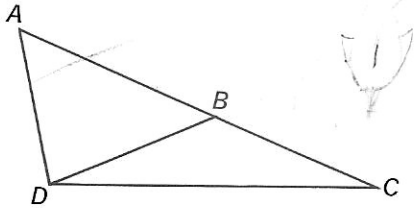


6. In the figure for Exercise 5, if $m\angle 7 = 71°$, what are measures of the other angles?

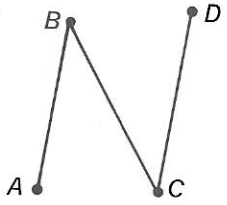
7. What are the measures of the angles of the parallelogram?



8. If $\overline{DB} \cong \overline{BC}$, $m\angle BDC = 23°$, and $m\angle A = 55°$, what is $m\angle ADC$?



9. If $\angle B \cong \angle C$ and all points are in the same plane, what can you say about $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$?

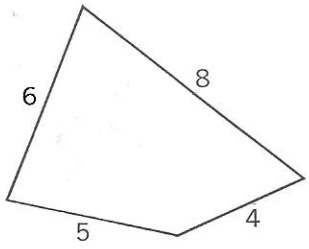


10. $\triangle ABC \cong \triangle DEF$ and $m\angle A = 37°$; $m\angle C = 74°$. What are measures of the angles of $\triangle DEF$?

11. $\triangle ABC$ is a right triangle with $\angle C$ the right angle. Give $m\angle B$ if

[a] $\overline{AC} \cong \overline{BC}$
[b] $m\angle A = 38°$
[c] $m\angle A = 2 \cdot m\angle B$

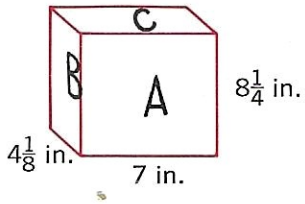
12. What is the sum of the measures of the angles of this quadrilateral?



13. What is the sum of the measures of the angles of a rhombus?

14. Find the area of

[a] face A
[b] face B
[c] face C



15. Find the total surface area of the block in Exercise 14.

16. Find the volume of the block in Exercise 14.

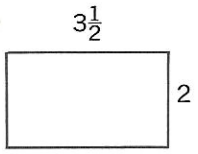
17. The length of a pipe is $72\frac{1}{4}$ in. to the nearest $\frac{1}{4}$ in. Write this measure, using the $\pm$ notation.

18. The legs of a right triangle have lengths of 12 and 5 cm. What is the square of the length of the hypotenuse?

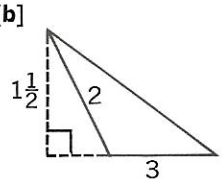
19. The legs of a right triangle are 6 and 8 cm. What is the length of the hypotenuse?

20. Find the area of each figure.

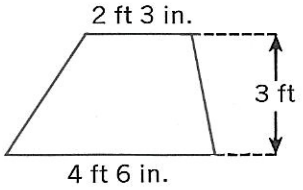
[a]



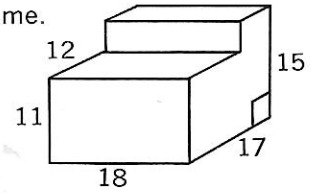
[b]



[c]



21. Find the volume.



CHAPTER SUPPLEMENT

Reading List

ADLER, IRVING. *The Giant Golden Book of Mathematics.* New York, Golden Press, 1960; pp. 33–38.

BERGAMINI, DAVID, and the Editors of *Life. Mathematics.* Life Science Library; New York; Time, Inc., 1963; pp. 38–50.

DUDENEY, H. E. *Amusements in Mathematics.* New York, Dover Publications, 1958; pp. 27–55.

GLENN, WILLIAM, and JOHNSON, DONOVAN. *The Pythagorean Theorem.* Exploring Mathematics on Your Own; St. Louis, Webster Publishing Company, 1960; pp. 1–18, 35–36.

MUIR, JANE. *Of Men and Numbers.* New York, Dell Publishing Company, 1961; pp. 7–34.

NORTON, M. SCOTT. *Basic Concepts of Vectors.* Exploring Mathematics on Your Own; St. Louis, Webster Publishing Company, 1963; pp. 2–23.

RAVIELLI, ANTHONY. *An Adventure in Geometry.* New York, Viking Press, 1957; pp. 99–105.

RUCHLIS, HY, and ENGELHARDT, JACK. *The Story of Mathematics—Geometry for the Young Scientist.* Irvington-on-Hudson, New York, Harvey House Publishing, 1958; pp. 86–110.

Activities

A. About 500 B.C., Pythagoras organized a secret society to discuss many issues of the day. Read and write about one mathematical issue which the Pythagoreans considered.

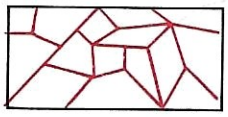
B. Find out about **Pythagorean Numbers** and be ready to explain them to the class.

C. There are many proofs of the Pythagorean Theorem. Find discussions about one of these and explain it to your class.

D. Construct several polyhedrons and use them to decorate your classroom. (See *The Story of Mathematics—Geometry for the Young Scientist*, pp. 86–110.)

E. Make up a problem which indicates how useful the Pythagorean Theorem is in practical measurement. Be able to show that the Theorem is a much more efficient or necessary method than attempting to measure directly (for example, measuring a distance with a large boulder obstructing the use of the tape measure).

F. Make a jig-saw puzzle by taking a rectangular piece of cardboard and coloring different shaped quadrilaterals on it. Then cut up the quadrilaterals and have a fellow classmate put your puzzle together.



CUMULATIVE REVIEW

1. Write each of the following as a power of ten.

[a] 1000
[b] 100,000
[c] 10,000,000,000

2. Find two numbers a and b so that

$$a + b = 26 \quad \text{and} \quad a - b = 8$$

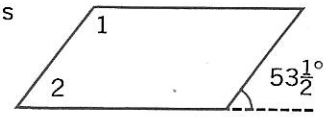
3. Solve the equations.

[a] $4 + x = 10\frac{2}{5}$
[b] $15\frac{2}{3} - w = 3\frac{1}{3}$
[c] $6 + (y \cdot 2) = 14$
[d] $\left(\frac{2}{5} \cdot 17\right) + \left(\frac{3}{5} \cdot 17\right) = r$

4. Write the numeral for the number of triangles in the set.

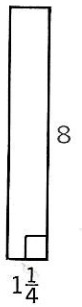
[a] In base ten
[b] In base five
[c] In base two

5. Find the measures for $\angle 1$ and $\angle 2$ in the parallelogram.

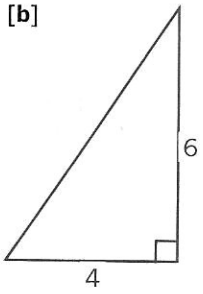


6. Find the areas.

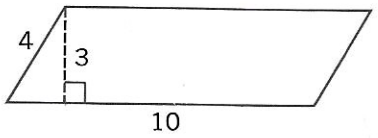
[a]



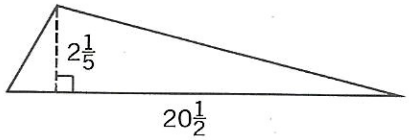
[b]



[c]

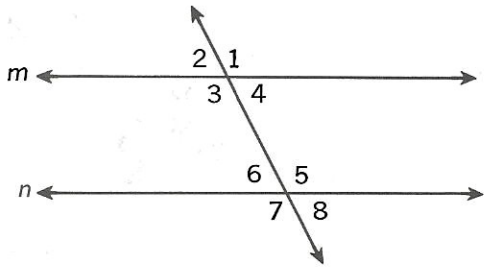


[d]



7. If two legs of a right triangle have lengths 3 and 4, what is the length of the hypotenuse?

8. Give the measure of every angle if $m \parallel n$ and the measure of $\angle 1$ is 117°.



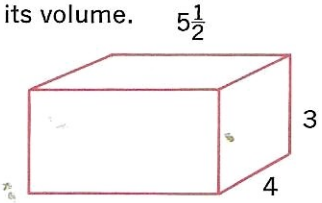
9. A (an) __?__ angle has a measure greater than 90°. (right, acute, obtuse)

10. Give the prime factorization of

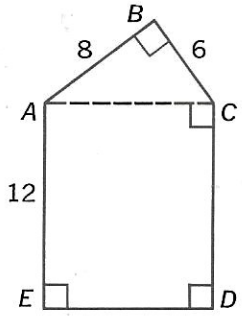
[a] 72 [b] 30 [c] 54 [d] 35

11. Find two numbers x and y so that $x \cdot y = 24$ and $x \div y = 6$.

12. [a] Find the surface area for the box.
[b] Find its volume.



13. Find the area of the region *ABCDE*.



14. A quadrilateral with all four sides congruent is a __?__. (trapezoid, rectangle, rhombus)

15. In each set, which fraction is not equivalent to any of the other fractions in the set?

[a] $\left\{\frac{1}{3}, \frac{3}{12}, \frac{5}{15}, \frac{8}{24}, \frac{6}{18}\right\}$

[b] $\left\{\frac{6}{8}, \frac{15}{20}, \frac{3}{4}, \frac{9}{16}, \frac{18}{24}\right\}$

16.

[a] function rule $\left(n + \frac{1}{2}\right) - \frac{1}{3}$

n	$f(n)$
2	$\frac{13}{6}$
$\frac{1}{2}$	
0	
$\frac{2}{5}$	
	1

[b] function rule $n \cdot 1\frac{1}{2}$

n	$f(n)$
2	3
1	
4	
	9
0	

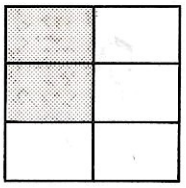
[c] function rule

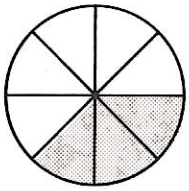
n	$f(n)$
5	$\frac{23}{5}$
$\frac{1}{2}$	$\frac{1}{10}$
$1\frac{2}{5}$	1
	$\frac{2}{7}$
2	

[d] function rule

n	$f(n)$
$1\frac{1}{3}$	8
$\frac{5}{3}$	10
$\frac{1}{4}$	$\frac{3}{2}$
	13
6	

17. What part of the figure is shaded?

[a] 

[b] 

18. Reduce to lowest terms

[a] $\frac{14}{35}$ [b] $\frac{15}{45}$ [c] $\frac{42}{63}$ [d] $\frac{26}{39}$

19. Give the correct sign ($<$, $=$, or $>$) for each ▒.

[a] $\frac{6}{7}$ ▒ $\frac{5}{6}$ [c] $\frac{1}{2}$ ▒ $\frac{5}{9}$

[b] $\frac{5}{8}$ ▒ $\frac{7}{12}$ [d] $\frac{11}{5}$ ▒ $\frac{13}{6}$

20. The GCF of 24 and 32 is

[a] 16 [b] 4 [c] 8 [d] 12

21. Give the sums.

[a] $1\frac{2}{3} + 2\frac{5}{6}$ [c] $2\frac{3}{4} + \frac{3}{5}$

[b] $\frac{5}{24} + \frac{7}{36}$ [d] $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

22. In each equation, what is the number for n?

[a] $\frac{3}{8} \cdot n = 1$ [c] $1\frac{1}{2} + \left(n \cdot \frac{1}{2}\right) = 3$

[b] $\frac{7}{5} \cdot \left(n + \frac{1}{7}\right) = 1$ [d] $(2 \cdot n) + 5 = 5 \cdot n$

23. Find the differences.

[a] $\frac{7}{9} - \frac{1}{6}$ [c] $2\frac{2}{3} - 1\frac{1}{4}$ [e] $6 - \frac{5}{11}$

[b] $5\frac{1}{4} - 3\frac{7}{8}$ [d] $7 - 3\frac{1}{5}$ [f] $\frac{5}{2} - \frac{5}{4}$

24. A car drove 50 mph for 2 hr 12 min. How far did it go?

25. Give the reciprocals for

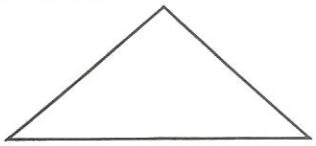
[a] $\frac{2}{7}$ [b] $2\frac{1}{3}$ [c] $\left(\frac{1}{2} + \frac{1}{5}\right)$

26. Solve the equations.

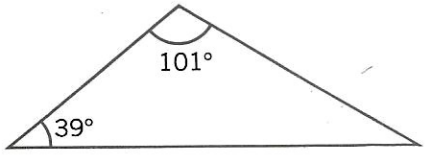
[a] $n \div 3 = 8$ [c] $4\frac{2}{3} \cdot 1\frac{1}{2} = n$

[b] $2\frac{1}{2} \cdot n = \frac{5}{12}$ [d] $3\frac{1}{3} \div n = \frac{5}{6}$

27. The perimeter of a triangle is $7\frac{3}{5}$. If two sides are $2\frac{1}{10}$ and $3\frac{4}{15}$, what is the length of the third side?

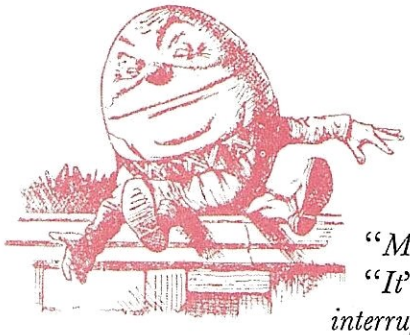


28. Give the measure of the third angle.



29. If a pipe has its length given as $83\frac{1}{2}$ in. $\pm$ $\frac{1}{8}$ in., between what limits does its length lie?

30. In a right triangle, the length of the hypotenuse is 17 in., and one leg is 15 in. What is the length of the other leg?



"*My* name *is Alice, but* ——"

"*It's a stupid name enough!*" *Humpty Dumpty interrupted impatiently.* "*What does it mean?*"

"Must *a name mean something?*" *Alice asked doubtfully.*

10

Decimals

10-1 NUMERALS FOR RATIONAL NUMBERS

Discussion Exercises

1. In Table A, fractions are used to give the batting averages (pct). In Table B, decimals are used. Which table gives you more help in comparing the batting averages?

A

Batting averages

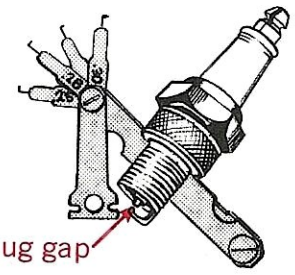
	g	ab	h	pct
Scott	153	586	190	$\frac{190}{586}$
Fisher	153	514	163	$\frac{163}{514}$

B

	g	ab	h	pct
Scott	153	586	190	.324
Fisher	153	514	163	.317

2. Figure C shows a gauge used to measure the gap in a spark plug. The decimals show the thickness of the metal. To show this information, why are decimals better than fractions?

C



3. The table at the right might be found in a science book.

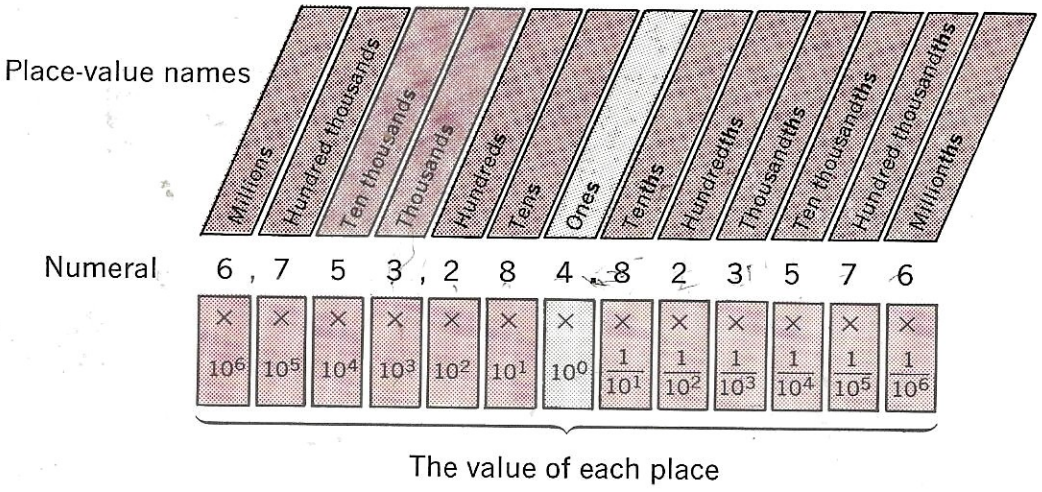
[a] To show this information, why are decimals better than fractions?

[b] Suppose you wanted to add numbers such as those in Table D. Can you see an advantage in using decimals?

D

Liquid	Pounds per gallon
Water	8.33
Gasoline	5.664
Kerosene	6.664
Mercury	113.288
Oil	7.497
Sulfuric acid	15.244
Sea water	8.58

To represent a rational number, a **decimal** is often used instead of a fraction. Decimals often make it easier to compare and compute with rational numbers. The diagram below shows how we extend our place-value system to give meaning to decimals. The dot between the ones and tenths place is a **decimal point.**



Discussion Exercises

1. Do you see any interesting relationships in the diagram above?

2. The 8 in the tens place represents $8 \cdot 10^1$. What does the 8 in the tenths place represent?

3. Tell what each digit in the numeral above represents.

Exercises

1. Copy the sentence. Give the missing word and number.

Example: 46.38. The 8 in the hundredths place represents $8 \cdot \frac{1}{10^2}$

[a] 476.835: The 4 in the __?__ place represents ▒.
[b] 476.835: The 3 in the __?__ place represents ▒.
[c] 9476.835: The 9 in the __?__ place represents ▒.
[d] 9476.835: The 5 in the __?__ place represents ▒.
[e] .4568: The 8 in the __?__ place represents ▒.
[f] 5.87654: The 4 in the __?__ place represents ▒.
[g] .123457: The 7 in the __?__ place represents ▒.

2. For Parts **a** through **e**, write each number in expanded notation (exponent form) as shown in Example A. For Parts **f** through **j**, write each number in expanded notation as shown in Example B.

(A) $634.265 = (6 \cdot 10^2) + (3 \cdot 10^1) + (4 \cdot 10^0) + \left(2 \cdot \frac{1}{10^1}\right) + \left(6 \cdot \frac{1}{10^2}\right) + \left(5 \cdot \frac{1}{10^3}\right)$

(B) $634.265 = 600 + 30 + 4 + \frac{2}{10} + \frac{6}{100} + \frac{5}{1000}$

[a] 35.42 [c] 562.34 [e] .0307 [g] 8.374 [i] 860.079
[b] 1.254 [d] 503.208 [f] 27.64 [h] 973.56 [j] .506

3. Give the decimal that represents each sum.

[a] $7 + \frac{3}{10}$
[b] $(6 \cdot 10) + (4 \cdot 1) + \left(3 \cdot \frac{1}{10}\right)$
[c] $(5 \cdot 1) + \left(3 \cdot \frac{1}{10}\right) + \left(7 \cdot \frac{1}{100}\right)$
[d] $5000 + 200 + 30 + 6 + \frac{3}{10} + \frac{7}{100} + \frac{5}{1000}$
[e] $8(100) + 5(10) + 3(1) + 2\left(\frac{1}{10}\right) + 7\left(\frac{1}{100}\right)$
[f] $7(10{,}000) + 5(1000) + 8(100) + 3(10) + 5(1) + 4\left(\frac{1}{10}\right) + 6\left(\frac{1}{100}\right) + 2\left(\frac{1}{1000}\right)$
[g] $8(100{,}000) + 3(100) + 7\left(\frac{1}{100}\right) + 5\left(\frac{1}{1000}\right)$
[h] $3(1{,}000{,}000) + 4(100{,}000) + 8(1) + 7\left(\frac{1}{10}\right) + 4\left(\frac{1}{10{,}000}\right)$
[i] $800 + 30 + 7 + \frac{2}{10} + \frac{3}{100} + \frac{5}{1000}$
[j] $1(10{,}000) + 1\left(\frac{1}{10{,}000}\right)$
[k] $(4 \cdot 100{,}000) + (3 \cdot 1000) + (7 \cdot 1) + \left(6 \cdot \frac{1}{10}\right) + \left(9 \cdot \frac{1}{100}\right)$
[l] $(3 \cdot 10{,}000) + (8 \cdot 1000) + (2 \cdot 100) + (3 \cdot 10) + 7 + \left(6 \cdot \frac{1}{10}\right) + \left(9 \cdot \frac{1}{100}\right)$
[m] $(6 \cdot 1000) + (4 \cdot 100) + (5 \cdot 10) + 1 + \left(5 \cdot \frac{1}{10}\right) + \left(4 \cdot \frac{1}{100}\right) + \left(6 \cdot \frac{1}{1000}\right)$
[n] $(7 \cdot 1{,}000{,}000) + (3 \cdot 1000) + 9 + \left(6 \cdot \frac{1}{100}\right) + \left(7 \cdot \frac{1}{1000}\right)$
[o] $(2 \cdot 100) + (6 \cdot 10) + 5 + \left(8 \cdot \frac{1}{10}\right) + \left(0 \cdot \frac{1}{100}\right) + \left(9 \cdot \frac{1}{1000}\right)$

4. Write each number, using expanded notation. Then find the sums as shown in the examples.

A $3.46 = 3 + \frac{4}{10} + \frac{6}{100} = 3 + \frac{40}{100} + \frac{6}{100} = 3\frac{46}{100}$

We read 3.46 as "three **point** four six" or "three **and** forty-six hundredths."

B $.759 = \frac{7}{10} + \frac{5}{100} + \frac{9}{1000} = \frac{700}{1000} + \frac{50}{1000} + \frac{9}{1000} = \frac{759}{1000}$

We read .759 as "**point** seven five nine" or "seven hundred fifty-nine thousandths."

[a] 7.52 [b] .364 [c] 9.452 [d] .2763 [e] 27.584

5. Write a decimal for each number represented.

[a] $8 + \frac{4}{10}$
[b] $21 + \frac{23}{100}$
[c] $4\frac{2}{100}$
[d] $0 + \frac{23}{100}$
[e] $1 + \frac{1}{1000}$
[f] $29 + \frac{472}{1000}$
[g] $614 + \frac{2152}{10,000}$
[h] $50\frac{47}{1000}$
[i] $50\frac{471}{10,000}$
[j] $0 + \frac{37,019}{100,000}$
[k] $4 + \frac{612}{1,000,000}$
[l] $25\frac{5607}{1,000,000}$

6. Write a mixed numeral for each number.

[a] 6.4
[b] 17.31
[c] 5.03
[d] .15
[e] 2.003
[f] 81.426
[g] 919.7154
[h] 40.0352
[i] .21746
[j] 3.000123
[k] 27.5680
[l] 286.11111

7. Give the correct sign (<, >, or =) for each ◯.

$\frac{89}{100} < \frac{98}{100}$ $\quad$ $\frac{67}{100} = \frac{670}{1000}$ $\quad$ $\frac{200}{1000} > \frac{199}{1000}$

[a] .89 ◯ .98
[b] .67 ◯ .670
[c] .2 ◯ .199
[d] 1.010 ◯ .999
[e] .54 ◯ .45
[f] 8.93 ◯ 8.930
[g] 5.001 ◯ 4.999
[h] .0988 ◯ .1
[i] .9 + .1 ◯ 1
[j] .009 + .1 ◯ .010
[k] .999 + .001 ◯ 1
[l] .09999 ◯ .1
[m] 5.654 ◯ 5.0654
[n] 98.999 ◯ 98.99
[o] .1 + .9 ◯ .9 + 1
[p] .99 + .99 ◯ .99 + .9
[q] .056 + 1 ◯ .06 + 1
[r] .7362 ◯ .73620

★ **8.** In decimal notation, the **ones** place is always just to the left of the decimal point. This is the usual agreement. Suppose, however, that we agree to write the decimal point **below** the ones place. Use our new system for the following questions.

[a] Write a phrase to show how you would read the decimal 532643.

[b] Give the decimal for the sum
$5(1000) + 2(100) + 8(10) + 7 + 8(\frac{1}{10}) + 2(\frac{1}{100}) + 5(\frac{1}{1000})$.

[c] Using decimals, write $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$.

[d] Using a decimal, write $1000 + \frac{1}{1000}$.

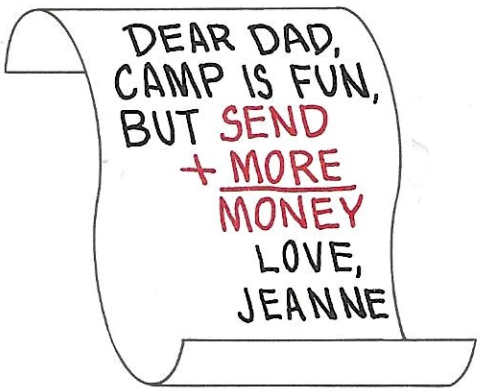
[e] Using a decimal, write $100 + \frac{1}{100}$.

[f] With the new agreement, in what ways is the decimal for one thousandth like the numeral for one thousand?

★ **9.** Write $.52_{(8)}$ in base ten notation.

Think

A camper sent her father the following message.



Can you help Jeanne's father figure out this addition puzzle in which SEND + MORE = MONEY? Each letter stands for a digit 0, 1, 2, . . . , 9, and no two letters may represent the same digit.

10-2 ADDING AND SUBTRACTING DECIMALS

Figure A shows the way one student worked the following problem.

$$23.42 + 32.5 + .671$$

A classmate checked the problem and marked it incorrect. Explain what the student who worked the problem did not understand.

A

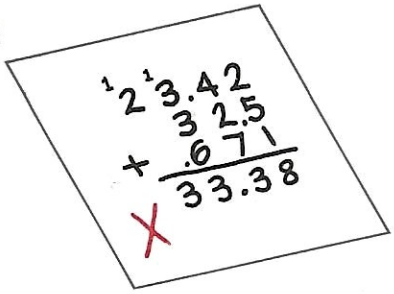
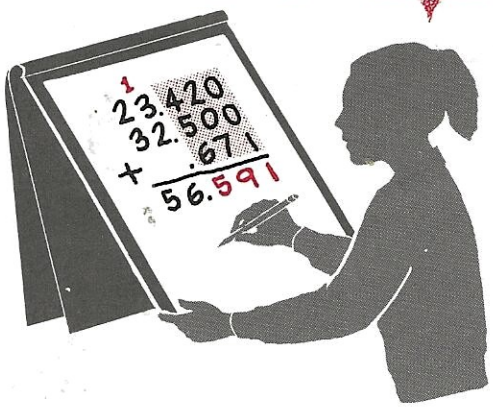


Figure B shows how you might think about the addition problem in Figure A.

Figure C shows how you might think about a subtraction problem using decimals.

B

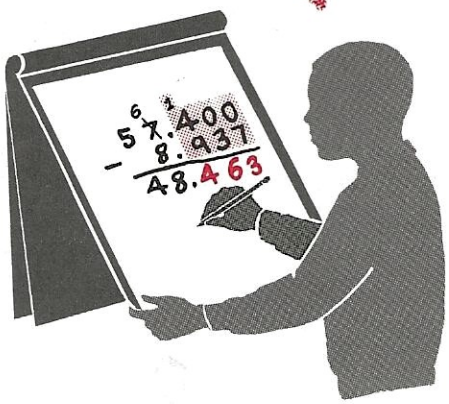
$$\frac{420}{1000} + \frac{500}{1000} + \frac{671}{1000} = 1\frac{591}{1000}$$



C

$$1\frac{400}{1000} = \frac{1400}{1000}$$

$$\frac{1400}{1000} - \frac{937}{1000} = \frac{463}{1000}$$



Discussion Exercises

1. Look at Figure B.
   - [a] Using fractions, explain why .42 can be written as .420.
   - [b] Using fractions, explain why .5 can be written as .500.
   - [c] Explain how the addition problem was completed.

2. Look at Figure C.
   - [a] Using fractions, explain why .4 can be written as .400.
   - [b] Explain why 1 (or $\frac{1000}{1000}$) must be "borrowed" from 57.
   - [c] Explain how the subtraction problem was completed.

Exercises

1. [a] 1 = _?_ tenths
[b] .1 = _?_ hundredths
[c] .01 = _?_ thousandths
[d] .001 = _?_ ten thousandths
[e] 5 = _?_ tenths
[f] .4 = _?_ hundredths
[g] .74 = _?_ thousandths
[h] .265 = _?_ ten thousandths

2. Use fractions to work each problem.

Examples: $.3 + .4 = \frac{3}{10} + \frac{4}{10} = \frac{7}{10} = .7$

$.3 + .24 = \frac{3}{10} + \frac{24}{100} = \frac{30}{100} + \frac{24}{100} = \frac{54}{100} = .54$

[a] .5 + .4 [c] .527 + .254 [e] .3 + .46 [g] .8 + .24 [i] .830 + .976
[b] .23 + .56 [d] .8 + .6 [f] .24 + .537 [h] .367 + .23 [j] .6 + .847

3. The decimal .50 is **equivalent** to the decimal .5. Write equivalent decimals **in hundredths** as needed and add. Do not use fractions.

Example: 1.4 + 2 + 3.14 →

$$\begin{array}{r} 1.40 \\ 2.00 \\ +3.14 \\ \hline 6.54 \end{array}$$

[a] 2.3 + 4 + 5.36 [c] 5 + 2.12 + .4 [e] .54 + 4.3 + .8
[b] 14.21 + .6 + 7.9 [d] 32.8 + .99 + 12 [f] 56.1 + 3.72 + 8.9

4. Find the sums without changing to hundredths.

Example: 7 + 1.4 + .62 →

$$\begin{array}{r} 7. \\ 1.4 \\ +.62 \\ \hline 9.02 \end{array}$$

[a] 8 + 2.3 + 1.54 [c] 8.7 + 14 + 1.62 [e] 4.6 + .25 + .8
[b] 3.5 + .62 + 11 [d] 17.04 + 1.3 + 25 [f] 9 + .63 + .2

5. Write equivalent decimals **in thousandths** as needed and add.

Example: 5.21 + .6 + 4.218 →

$$\begin{array}{r} 5.210 \\ .600 \\ +4.218 \\ \hline 10.028 \end{array}$$

[a] 1.412 + 86 + 2.3 [c] 2.3 + 17 + 4.265 [e] 8.62 + .9 + .213
[b] 76.2 + .34 + .079 [d] 13.8 + 7.243 + 11.63 [f] 47.54 + 92.76 + 8.7

6. Write equivalent decimals in **thousandths** as needed and subtract.

Example: 36.47 − 4.293 → 36.470 − 4.293 = 32.177

[a] 49.5 − .047 [b] 58.267 − .19 [c] 92 − .143 [d] 3.06 − .999

7. Compute the following sums and differences.

[a] .48 + .31
[b] 3.2 + 4.7
[c] .5 + 3 + .02
[d] 2.54 + 37.3
[e] 13.354 + 423.4568
[f] .01 − .001

[g] 4.763 − .321

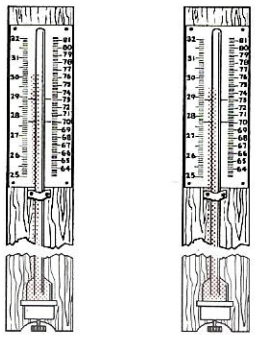
[h] 6.563 + .278 + 37.937

[i] 5.346 + 35.27 + 161.2 + 4.387

[j] 146.2 − 21.462
[k] 1.002 + .0102
[l] 1.05 + .75 + 21.5
[m] (3.1416 + 1.4142) − 2.8
[n] (.306 − .21) + .067
[o] 10.2 + .34 + 4
[p] 100 − .01
[q] 90 + 9 + .9 + .09

SHORT STORIES

1. Barometer reading today: 30.05 in. Barometer reading yesterday: 29.47 in. How much did it rise?



2. Rectangular field: length, 236.29 m; width, 98.86 m. Find the perimeter.

3. Weights per cubic foot: gold, 1204.3; lead, 705.1; copper, 554.74. A cubic foot of lead with a cubic foot of copper weighs how much more than a cubic foot of gold?

4. Travel. Monday: 463.8 mi. Tuesday: 579.4 mi. Wednesday: 624.6 mi. How far in three days?

TRAVEL RECORD

Day	Miles	Gas ($)	
Mon.	463.8	6	83
Tues.	579.4	8	39
Wed.	624.6	9	52

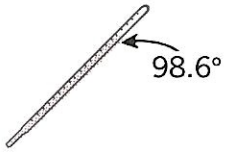
5. Average brain weights: man, 38.8 oz; woman, 37.04 oz. How much more does a man's brain weigh?

6. Satellite: takes 95.57 min to orbit the earth.

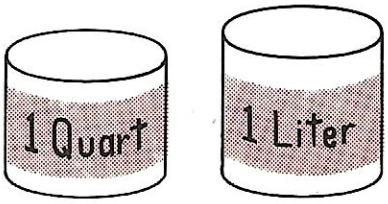
[a] How many minutes for two orbits?
★ [b] About how many orbits in two 24-hr days?

PROBLEMS

1. The normal body temperature of a man is 98.6° (Fahrenheit). Often, infection causes the temperature to rise. A high fever may raise the normal body temperature 5.8°. How high would this be?



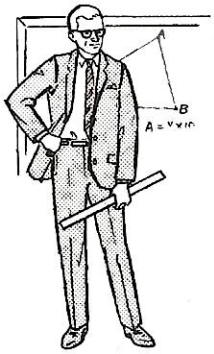
2. The volume of a 1-qt container is 57.75 cu in. The volume of a 1-liter (ℓ) container is 61.025 cu in. How much greater is the volume of the liter than the volume of the quart?



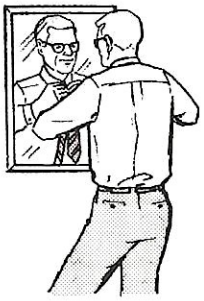
3. Energy is measured in units called **calories**, the heat producing value of food. The number of calories used per minute for various activities are listed below.

Lying at rest: 1.19

Walking: 5.1



Standing: 1.25



Washing and dressing: 3.56

How many calories does a man use if he rests two minutes, washes and dresses in two minutes, stands two minutes, and walks for two minutes?

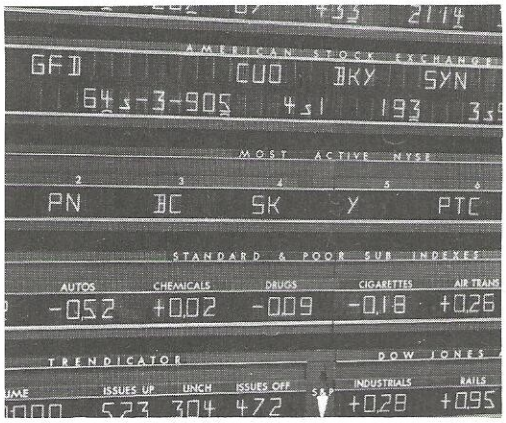
4. The body of an average man contains about 1.5 gal of blood. The body of an average woman contains about .875 gal. How much greater is the blood capacity of a man?

Think

Solve the equation:
$10 + n = 10 \cdot n$

10-3 MULTIPLYING DECIMALS

Decimals are used as abbreviations for fractions with denominators that are powers of ten. If you can multiply using these fractions, then you will find decimal multiplication easy to understand. In multiplication, it will often be helpful to describe decimals as shown in the diagram below.



Decimals are frequently used in stock notation.

Tenths		Hundredths		Thousandths	
.3	one-place	.54	two-place	.124	three-place
8.6	decimals	.08	decimals	.050	decimals
34.5		5.99		8.007	

. . .

Exercises

1. Give the fraction for ***a***, the decimal for ***b***, and the correct word for the blank.

[a] $\frac{1}{10} \times \frac{1}{10} = a$

$.1 \times .1 = b$

tenths $\times$ tenths $=$ __?__

[b] $\frac{1}{10} \times \frac{1}{100} = a$

$.1 \times .01 = b$

tenths $\times$ hundredths $=$ __?__

[c] $\frac{1}{10} \times \frac{1}{1000} = a$

$.1 \times .001 = b$

tenths $\times$ thousandths $=$ __?__

[d] $\frac{1}{100} \times \frac{1}{100} = a$

$.01 \times .01 = b$

hundredths $\times$ hundredths $=$ __?__

2. Study the example. Then find the products.

Think:

8 × 3 = 24

tenths × tenths = hundredths

.8 × .3 = **P** **P = .24**

[a] .4 × .3
[b] .5 × .4
[c] .8 × .6
[d] .9 × .8
[e] .5 × .8
[f] .6 × .3
[g] .9 × .1
[h] .5 × .5
[i] .9 × .9

3. Study the example. Then find the products.

Think:

4 × 9 = 36

tenths × hundredths = thousandths

.4 × .09 = **P** **P = .036**

[a] .5 × .07
[b] .9 × .08
[c] .8 × .06
[d] .8 × .96
[e] .09 × .7
[f] .47 × .4
[g] .6 × .25
[h] .52 × .3
[i] .9 × .06

4. Study the example. Then find the products.

Think:

6 × 3 = 18

tenths × thousandths = ten thousandths

.6 × .003 = **P** **P = .0018**

[a] .4 × .007
[b] .4 × .027
[c] .4 × .527
[d] .326 × .8
[e] .9 × .406
[f] .7 × .300
[g] .425 × .3
[h] .8 × .946

5. Study the example. Then find the products.

Think:

42 × 27 = 1134

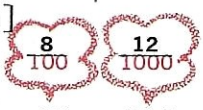
hundredths × hundredths = ten thousandths

.42 × .27 = **P** **P = .1134**

[a] .34 × .16
[b] .05 × .24
[c] .08 × .07
[d] .49 × .76
[e] .27 × .09
[f] .25 × .04
[g] .08 × .51
[h] .25 × .25

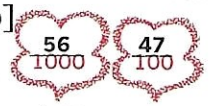
6. Find the products.

[a]



.08 × .012 = **P**

[b]



.056 × .47 = **P**

[c]

.237 × .359 = **P**

7. Multiplying numbers represented by decimals is much like multiplying whole numbers. Study the examples. Then find the products.

A First find 76 × 3, then think:

7.6 tenths (one-place decimal)
times
×.3 tenths (one-place decimal)
equals
228 hundredths (two-place decimal)

Place decimal point here.

B First find 942 × 31, then think:

94.2 tenths (one-place decimal)
times
×.31 hundredths (two-place decimal)
equals
942
2826
29202 thousandths (three-place decimal)

Place decimal point here.

[a] 56.3 ×2	[b] 43.5 ×.2	[c] 3.6 ×4.5	[d] 4.5 ×9.8	[e] .15 ×66
[f] .68 ×1.8	[g] 5.9 ×.77	[h] .41 ×.26	[i] .507 ×4.9	[j] .379 ×47.6

8. Complete each sentence.

[a] When each factor is named by a one-place decimal, the product is named by a __?__-place decimal.

[b] When one factor is named by a one-place decimal and the other by a two-place decimal, the product is named by a __?__-place decimal.

[c] When each factor is named by a two-place decimal, the product is named by a __?__-place decimal.

[d] To find the number of decimal places in the product, we __?__ the number of decimal places in the two factors.

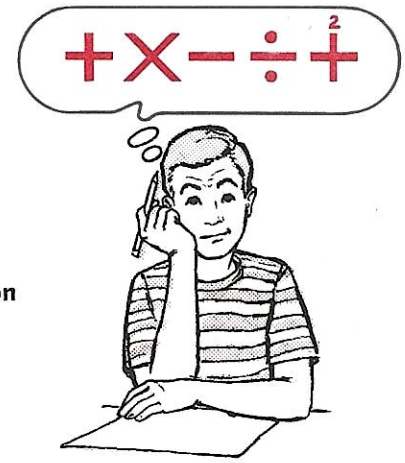
9. Find the products.

[a] 4.67 ×.394	[b] 8.462 ×.76	[c] 9.763 ×.07	[d] .346 ×.571	[e] .408 ×.930

★**10.** For each problem, find one number that can serve as both quotient and divisor.

[a] ▮▮▮) 2.25 [b] ▮▮▮) 6.25 [c] ▮▮▮) 590.49 [d] ▮▮▮) 17.0569

Think



You are familiar with the four operations, **addition**, **multiplication**, **subtraction**, and **division** as well as the **basic principles** for addition and multiplication. Just for fun, let's invent a new operation called "squareaddition." Study the examples to figure out how to "squareadd." We might call the answer to a squareaddition problem a "squm."

Examples
$2 \overset{2}{+} 1 = 5$
$3 \overset{2}{+} 4 = 25$
$10 \overset{2}{+} 10 = 200$

1. Find the "squms."

[a] $3 \overset{2}{+} 1$ [c] $7 \overset{2}{+} 7$ [e] $1 \overset{2}{+} 1$

[b] $4 \overset{2}{+} 5$ [d] $15 \overset{2}{+} 25$ [f] $9 \overset{2}{+} 0$

2. Do the zero, the commutative, and the associative principles hold true for "squareaddition"? Give an example for each.

3. Does the distributive (multiplication-squareaddition) principle hold true? Give an example.

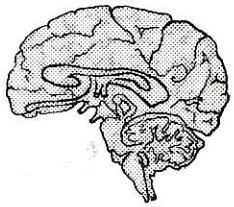
4. Find as many pairs of whole numbers as you can whose squm is some whole number squared.

5. Invent a new operation of your own. See if a classmate can figure it out. Check to see if the basic principles hold true.

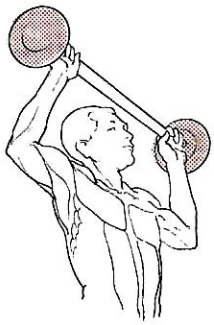
SHORT STORIES

1. Amount of water in body: .58 times the body weight. Body weight, 152.5 lb. How many pounds of water?

2. Man's brain: .03 of body weight. Body weight: 152.5 lb. Brain weight?

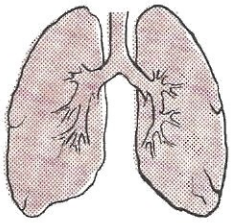


3. Man's muscles: .4 of body weight. Body weight: 152.5 lb. Muscle weight?



4. We breathe 400 cu ft of air per day. Oxygen into blood: .05 of total air. How many cubic feet of oxygen into blood each day?

5. Woman reading: .358 qt of air into lungs each breath, 22 breaths per minute. How many gallons of air in 1 min (to nearest gallon)?



6. Woman, calories used per minute: lying at rest, .98; standing, 1.11; peeling potatoes, 1.29; making beds, 5.4.
[a] Resting, 15 min. Peeling potatoes, 8 min. Making beds, 12 min. How many calories?
[b] Standing in line, 24.5 min. How many calories?

7. Man washing dishes, 3.3 calories per minute. Woman washing dishes, 1.53 calories per minute. Man finishes in 8 min. Woman finishes in 17 min.
[a] Who uses the most calories?
[b] How many more?

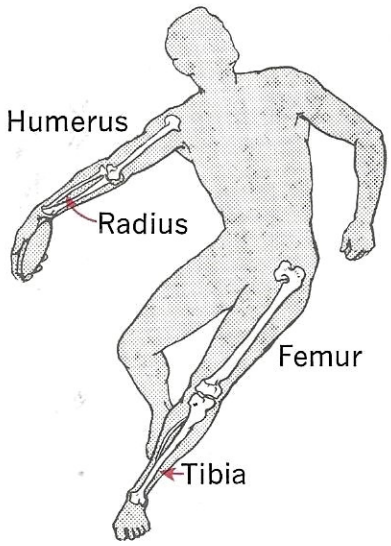
8. One qt, 57.75 cu in. Lung capacity of average man, 6.76 qt. Lung capacity, how many cubic inches?

9. Man, 18 breaths per minute. When resting, .793 qt of air each breath. When doing light work, 1.77 qt. When doing heavy work, 2.15 qt. Rested 15 min. Worked lightly 15 min. Worked hard 15 min.
[a] How many quarts of air used?
[b] How many cubic inches of air? (1 qt, 57.75 cu in.)
[c] How many cubic feet of air?

10. Human bones: .18 of body weight. Body weight: 152.5 lb. Weight of bones?

11. Heart: pumps 12 gal per min during strenuous exercise. During relaxation, .125 as many gallons. How many gallons?

PROBLEMS—ESTIMATING BODY HEIGHTS



Anthropologists, using a single bone from a human skeleton, can estimate quite accurately the height of a man or woman who lived many centuries ago. For example, if an anthropologist finds an 11-in. radius from a caveman, he can estimate the height of the caveman as follows:

$$(3.271 \times 11) + 33.829$$

or 69.810 in. Thus, the man was about 5 ft 10 in. tall.

1. An 18-in. femur was found. About how tall was the man?

2. The length of a humerus bone from a woman who lived 20,000 years ago is 12.7 in. What would be a good estimate of the woman's height (to the nearest inch)?

3. The skeleton of a Neanderthal man who lived about 50,000 years ago contained a tibia 15 in. long. Give an estimate of the man's height.

Height (inches)	
Male	
(2.894 × length of humerus)	+ 27.811
(3.271 × length of radius)	+ 33.829
(1.880 × length of femur)	+ 32.010
(2.376 × length of tibia)	+ 30.970
Female	
(2.754 × length of humerus)	+ 28.140
(3.343 × length of radius)	+ 31.978
(1.945 × length of femur)	+ 28.679
(2.352 × length of tibia)	+ 29.439

★ **4.** A man is 71.362 in. tall. Using the table, accurately estimate the length of the bone from the knee joint to the ankle joint.

★ **5.** Measure, as accurately as you can, the length of the radius and humerus of an adult and check the accuracy of the table.

★ **6.** A student's height was found to be the same as the distance from fingertip to fingertip when his arms were outstretched. How much does your height differ from this distance?

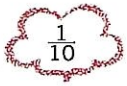
★ **7.** One student's height in inches was 8.7 times the length of his middle finger measured to the nearest tenth centimeter. How much does your height differ from this distance?

10-4 MULTIPLYING WITH POWERS OF TEN

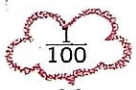
In order to estimate with decimals, it is important to be able to quickly multiply a decimal by 10, 100, 1000, as well as by .1, .01, .001, and so forth. The exercises will help you understand how to do this, and they will suggest a shortcut for finding these products.

Exercises

1. Solve the equations. You may find it helpful to think about fractions.



[a] $10 \times .1 = a$
[b] $10 \times .2 = b$
[c] $10 \times .3 = c$
[d] $10 \times .7 = d$



[e] $10 \times .01 = e$
[f] $10 \times .02 = f$
[g] $10 \times .03 = g$
[h] $10 \times .09 = h$

[i] $10 \times .001 = i$
[j] $10 \times .002 = j$
[k] $10 \times .005 = k$
[l] $10 \times .008 = l$

2. Study the example. Then copy and complete each exercise.
Using the distributive principle,

$$10 \times 5.34 = (10 \times 5) + (10 \times .3) + (10 \times .04)$$

Since $\left\{\begin{array}{l} 10 \times 5 = \mathbf{50} \\ 10 \times .3 = \mathbf{3} \\ 10 \times .04 = \mathbf{.4} \end{array}\right\}$ we know that $10 \times 5.34 = \mathbf{53.4}$.

[a] $\left.\begin{array}{l} 10 \times 6 = a \\ 10 \times .7 = b \end{array}\right\} 10 \times 6.7 = c$

[b] $\left.\begin{array}{l} 10 \times .3 = a \\ 10 \times .08 = b \end{array}\right\} 10 \times .38 = c$

[c] $\left.\begin{array}{l} 10 \times 4 = a \\ 10 \times .5 = b \\ 10 \times .07 = c \end{array}\right\} 10 \times 4.57 = d$

[d] $\left.\begin{array}{l} 10 \times .8 = a \\ 10 \times .06 = b \\ 10 \times .002 = c \end{array}\right\} 10 \times .862 = d$

[e] $\left.\begin{array}{l} 10 \times 29 = a \\ 10 \times .08 = b \\ 10 \times .007 = c \end{array}\right\} 10 \times 29.087 = d$

3. To write the product of 10 and a number named by a decimal, rewrite the decimal point __?__ places farther to the right.

4. Find the products.

[a] 10×8.5
[b] $10 \times .64$
[c] 10×9.037
[d] 10×45.62
[e] $.496 \times 10$
[f] 8.437×10
[g] 9.054×10
[h] 10×57.3
[i] 6.49×10
[j] $.037 \times 10$
[k] 10×32.76
[l] 10×10.10

5. Since $100 = 10 \times 10$, you can multiply a number by 100 simply by multiplying by 10 twice. Find the products.

[a] $100 \times 4.39 = 10 \times (10 \times 4.39) = P$
[b] $100 \times .567 = 10 \times (10 \times .567) = P$
[c] $100 \times .8 = 10 \times (10 \times .8) = P$
[d] $100 \times 54.3 = 10 \times (10 \times 54.3) = P$
[e] $100 \times .076 = 10 \times (10 \times .076) = P$
[f] $4.7 \times 100 = (4.7 \times 10) \times 10 = P$
[g] $.06 \times 100 = (.06 \times 10) \times 10 = P$
[h] $9.346 \times 100 = (9.346 \times 10) \times 10 = P$
[i] $.004 \times 100 = (.004 \times 10) \times 10 = P$
[j] $5.43 \times 100 = (5.43 \times 10) \times 10 = P$

6. To write the product of 100 times a number named by a decimal, rewrite the decimal point __?__ places to the right.

7. Find the products.

[a] 100×5.03
[b] $100 \times .27$
[c] 100×29.5
[d] $100 \times .937$
[e] 8.462×100
[f] $100 \times .762$
[g] 8.34×100
[h] 19.6×100

8. Since $1000 = 10 \times 10 \times 10$, you can multiply by 1000 simply by multiplying by 10 three times. Find each product.

[a] $1000 \times .376 = 10 \times 10 \times (10 \times .376) = P$
[b] $1000 \times .84 = 10 \times 10 \times (10 \times .84) = P$
[c] $1000 \times 52.4 = 10 \times 10 \times (10 \times 52.4) = P$
[d] $1000 \times .008 = 10 \times 10 \times (10 \times .008) = P$
[e] $5.623 \times 1000 = (5.623 \times 10) \times 10 \times 10 = P$
[f] $6.97 \times 1000 = (6.97 \times 10) \times 10 \times 10 = P$
[g] $.017 \times 1000 = (.017 \times 10) \times 10 \times 10 = P$
[h] $.3426 \times 1000 = (.3426 \times 10) \times 10 \times 10 = P$

9. To write the product of 1000 and a number named by a decimal, rewrite the decimal point __?__ places to the right.

10. Find the products.

[a] 1000×5.468
[b] 1000×3.14
[c] 8.2×1000
[d] 8.407×1000

11. Find the products. You may find it helpful to think about fractions.

[a] $.1 \times 7$
[f] $.1 \times 2$
[k] $.1 \times 10$

[b] $.1 \times 28$
[g] $14 \times .1$
[l] $100 \times .1$

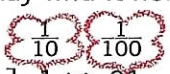
[c] $.1 \times 346$
[h] $20 \times .1$
[m] $367 \times .1$

[d] $8469 \times .1$
[i] $500 \times .1$
[n] $42 \times .1$

[e] $.1 \times 400$
[j] $.1 \times 1$
[o] $.1 \times 8673$

12. Solve the equations. You may find it helpful to think about fractions.

[a] $.1 \times .1 = a$
[b] $.1 \times .2 = b$
[c] $.1 \times .5 = c$
[d] $.1 \times .9 = d$



[e] $.1 \times .01 = e$
[f] $.1 \times .03 = f$
[g] $.1 \times .06 = g$
[h] $.1 \times .08 = h$

[i] $.1 \times .001 = i$
[j] $.1 \times .004 = j$
[k] $.1 \times .007 = k$
[l] $.1 \times .009 = l$

13. Study the example. Then copy and complete each exercise.

$$.1 \times 7.24 = (.1 \times 7) + (.1 \times .2) + (.1 \times .04)$$

Since $\left\{\begin{matrix} .1 \times 7 & = .7 \\ .1 \times .2 & = .02 \\ .1 \times .04 & = .004 \end{matrix}\right\}$ we know that $.1 \times 7.24 = .724$.

[a] $\left.\begin{matrix} .1 \times 2 & = a \\ .1 \times .3 & = b \\ .1 \times .04 & = c \end{matrix}\right\} .1 \times 2.34 = d$

[b] $\left.\begin{matrix} .1 \times 24 & = a \\ .1 \times .08 & = b \\ .1 \times .003 & = c \end{matrix}\right\} .1 \times 24.083 = d$

[c] $\left.\begin{matrix} .1 \times 592 & = a \\ .1 \times .8 & = b \\ .1 \times .04 & = c \end{matrix}\right\} .1 \times 592.84 = d$

[d] $\left.\begin{matrix} .1 \times .3 & = a \\ .1 \times .02 & = b \\ .1 \times .004 & = c \end{matrix}\right\} .1 \times .324 = d$

14. To write the product of .1 and a number named by a decimal, rewrite the decimal point __?__ places to the left.

15. Find the products.

[a] $.1 \times 8.3$ [b] $.1 \times 25.17$ [c] $.1 \times 432.08$ [d] $156.2 \times .1$ [e] $.83 \times .1$

16. Since $.01 = .1 \times .1$ and $.001 = .1 \times .1 \times .1$, you can multiply a number by .01 by multiplying by .1 twice, and you can multiply a number by .001 by multiplying by .1 three times. Find the products.

[a] $.01 \times .32 = .1 \times (.1 \times .32) = P$
[b] $.01 \times 9.3 = .1 \times (.1 \times 9.3) = P$
[c] $.01 \times 57 = .1 \times (.1 \times 57) = P$
[d] $.01 \times 69.3 = .1 \times (.1 \times 69.3) = P$
[e] $.001 \times 5946.5 = .1 \times .1 \times (.1 \times 5946.5) = P$
[f] $.001 \times 23.7 = .1 \times .1 \times (.1 \times 23.7) = P$
[g] $.001 \times .8 = .1 \times .1 \times (.1 \times .8) = P$
[h] $.001 \times 5.63 = .1 \times .1 \times (.1 \times 5.63) = P$

17. [a] To write the product of .01 and a number named by a decimal, rewrite the decimal point __?__ places to the left.

[b] To write the product of .001 and a number named by a decimal, rewrite the decimal point __?__ places to the left.

18. Give the missing numbers for each row in the table.

	× 1000	× 100	× 10	Number	× .1	× .01	× .001
[a]				3.71			
[b]				42.193			
[c]				.04			
[d]				10,010.0110			

19. Joe discovered an easy way to multiply by powers of ten. He made this chart for his classmates to complete. Complete the chart.

[a] To multiply a decimal by 10^1, move the decimal point 1 place to the right.

[b] To multiply a decimal by 10^2, move the decimal point ▒ places to the __?__.

[c] To multiply a decimal by 10^3, move the decimal point ▒ places to the __?__.

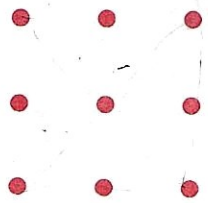
[d] To multiply a decimal by $\frac{1}{10^1}$, move the decimal point ▒ places to the __?__.

[e] To multiply a decimal by $\frac{1}{10^2}$, move the decimal point ▒ places to the __?__.

[f] To multiply a decimal by $\frac{1}{10^3}$, move the decimal point ▒ places to the __?__.

Think

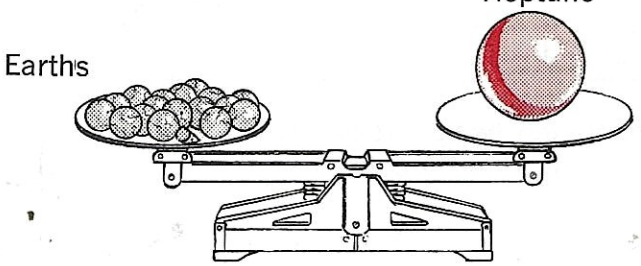
Draw an array of nine dots like the one to the right.



Place your pencil at one dot and, without lifting your pencil from the paper and without retracing, draw four line segments so that each dot lies on one of the segments.

10-5 APPROXIMATION (ROUNDING) AND ESTIMATION

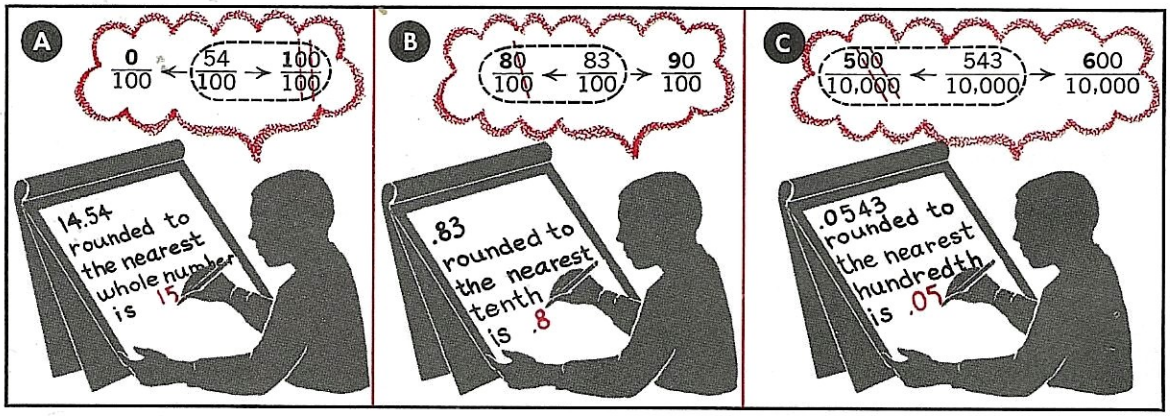
The table compares the mass of the planets with the mass of the earth. When we say that the **mass** of Neptune is 17.20 compared with the mass of earth, you might think of a giant balance



Mass of each planet compared with earth	
Earth	1.000
Mercury	.0543
Mars	.1069
Venus	.8136
Pluto	.8300
Uranus	14.54
Neptune	17.20
Saturn	95.3
Jupiter	318.35

in which 17.20 "earths" were needed to balance 1 "Neptune."

The mass comparisons given in the table are approximately correct. Often, when it is desirable to give even less accurate information, we might round the numbers in the table to the nearest whole number, the nearest tenth, or the nearest hundredth. Study the examples.


A
0/100 ← 54/100 → 100/100
14.54 rounded to the nearest whole number is 15
B
80/100 ← 83/100 → 90/100
.83 rounded to the nearest tenth is .8
C
500/10,000 ← 543/10,000 → 600/10,000
.0543 rounded to the nearest hundredth is .05

Rounding to the **nearest whole number**

Rounding to the **nearest tenth**

Rounding to the **nearest hundredth**

Discussion Exercises

1. In Example A, explain why the student chose the whole number 15 instead of 14.
2. In Example B, explain why the answer was .8 and not .9.
3. In Example C, explain why the number was rounded to .05 and not .06.

Exercises

1. Choose the number that best completes the sentence. **When a number is halfway between two numbers, we round to the higher number.**

[a] 6.57 rounded to the nearest whole number $\longrightarrow$ {6, 7}

[b] .849 rounded to the nearest tenth $\longrightarrow$ {.8, .9}

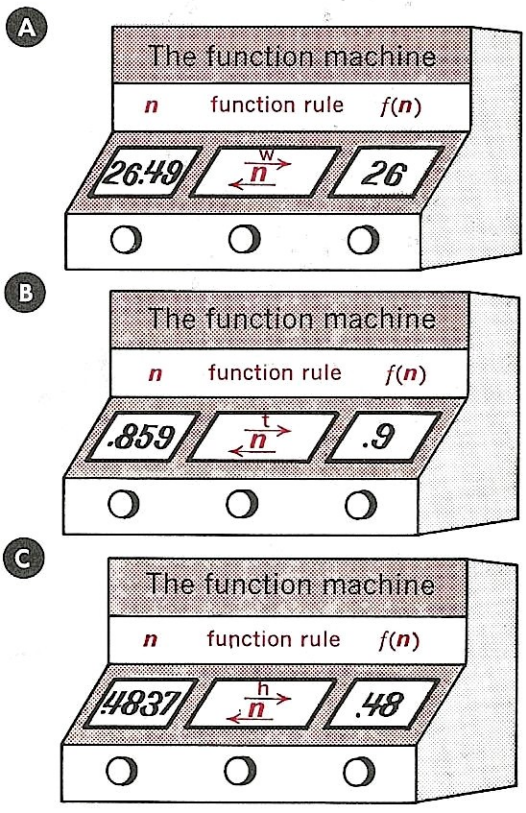
[c] 4.6468 rounded to the nearest hundredth $\longrightarrow$ {4.64, 4.65}

[d] .864327 rounded to the nearest hundredth $\longrightarrow$ {.86, .87}

[e] .327496 rounded to the nearest thousandth $\longrightarrow$ {.327, .328}

[f] .25 rounded to the nearest tenth $\longrightarrow$ {.2, .3}

2. Fred thought of some special functions. Since he could not use +, −, ×, or ÷ to describe his function rules, he used the symbols $\overset{h}{\underset{}{\rightarrow}}\ n\ \underset{}{\leftarrow}$, $\overset{t}{\underset{}{\rightarrow}}\ n\ \underset{}{\leftarrow}$, and $\overset{w}{\underset{}{\rightarrow}}\ n\ \underset{}{\leftarrow}$. Study the machines and table in order to understand Fred's symbols. Then complete the table.



	function rule $\overset{w}{\underset{}{\rightarrow}}\ n\ \underset{}{\leftarrow}$	
	n	$f(n)$
	5.7	6
	29.43	29
[a]	347.299	
[b]	34.53	
[c]	29.5	

3. Use the function rules of Exercise 2 for the following problems.

function rule $\xrightarrow[n]{w}$

	n	$f(n)$
[a]	598.7	
[b]	47.634	
[c]	5.4999	
[d]	8.50010	
[e]	56.3742	
[f]	7.0999	

function rule $\xrightarrow[n]{t}$

	n	$f(n)$
	.842	.8
	.3513	.4
[g]	5.749	
[h]	29.362	
[i]	.85106	
[j]	.35	

function rule $\xrightarrow[n]{h}$

	n	$f(n)$
	.3172	.32
	5.746	5.75
[k]	.06499	
[l]	.196	
[m]	54.37652	
[n]	.435	

4. Study the example. For each problem, give the whole numbers for ***a*** and ***b***. Then give the estimate ***c***.

To estimate → round each factor to the **nearest whole number.**

Find or estimate the whole number product.

4.73×8.49

$a \times b$ $\quad a = 5,\ b = 8$

c $\quad c = 40$

4.73×8.49 is approximately 40.

[a] 5.9×4.67 $\quad a \times b = c$

[c] 5.697×9.83 $\quad a \times b = c$

[e] 7.85×29.87 $\quad a \times b = c$

[g] 59.42×98.943 $\quad a \times b = c$

[b] 8.3×7 $\quad a \times b = c$

[d] 7×30.9573 $\quad a \times b = c$

[f] 9.39×29.32 $\quad a \times b = c$

[h] 51.094×202.78 $\quad a \times b = c$

5. Study the example. For each problem, give the one-place decimal for ***a*** and the whole number for ***b***. Then give the estimate ***c***.

To estimate → $.348 \times 9.24$
round .348 to the **nearest tenth**; round 9.24 to the **nearest whole number**.
Find or estimate the one-place decimal.

$a \times b = c$

$a = .3,\ b = 9$

$c = 2.7$

$.348 \times 9.24$ is approximately 2.7.

[a] $.297 \times 8$ [b] $.189 \times 7.34$ [c] $.4297 \times 9$ [d] $.5013 \times 5.23$

6. Study the example. For each problem, give the two-place decimal for ***a*** and the whole number for ***b***. Then give the estimate ***c***.

To estimate → $.0576 \times 7$
round .0576 to the **nearest hundredth**.
Find or estimate the two-place decimal.

$a \times b = c$

$a = .06,\ b = 7$

$c = .42$

$.0576 \times 7$ is approximately equal to .42.

[a] $.0348 \times 7$ [b] $.02507 \times 9$ [c] $.03419 \times 8$ [d] $.07395 \times 6.29$

7. The mass of the earth is about seven sextillion tons. The table on page 314 will give you the other information you need to complete these exercises.

[a] Round the appropriate numbers in the table to the nearest whole number and estimate the mass of Jupiter.

[b] Round the appropriate numbers in the table to the nearest hundredth and estimate the mass of Mars.

[c] Round the appropriate numbers in the table to the nearest whole number and estimate the mass of Saturn.

[d] Round the appropriate numbers in the table to the nearest tenth and estimate the mass of Venus.

[e] Round the appropriate numbers in the table to the nearest hundredth and estimate the mass of Mercury.

8. Estimate the products.

[a] 10.001×54.39
[b] 10.1×99.98
[c] 10.113×153.71
[d] $.512 \times 10.0037$
[e] $100.11 \times .5032$
[f] $102 \times .0305$
[g] 45.21×98.532
[h] 5.324×1004.376
[i] 45.113×20.0974
[j] 3.5432×9998.879
[k] $.1 \times 59.91$
[l] $.101 \times 59.91$
[m] $.201 \times 59.91$
[n] $.021 \times 48.04$
[o] $.021 \times 486.31$
[p] $.009 \times 462$

★ **9.** Which of the numbers is the best approximation of $.00217 \times .3596$?

[a] .7812 [b] .07812 [c] .007812 [d] .0007812 [e] 7.812000

★ **10.** Which of the numbers is the best approximation of $4.52 \div .423$?

[a] 1.068 [b] 10.68 [c] 106.8 [d] .01068 [e] 1068

(**Hint**: If the product is 4.52 and one factor is .423, would the missing factor be close to .01, .1, 1, 10, 100, or 1000?)

★ **11.** Which of the numbers is the best approximation of $3.251 \div .0234$?

[a] 1.3893 [b] 13.893 [c] 138.93 [d] 1389.3 [e] .013893

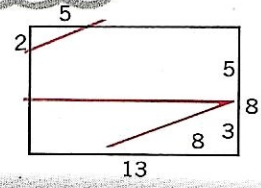
★ **12.** Give the missing numerators ***a***, ***b***, ***c***, and ***d***.

$$\frac{1}{7} = \frac{a}{105} = \frac{b}{1001} = \frac{c}{10{,}003} = \frac{d}{100{,}002}$$

Use these fractions to give four decimal approximations of $\frac{1}{7}$.
Prove that each of these approximations is more than $\frac{1}{7}$.

★ **13.** Use the method of Exercise 12 to get three decimal approximations of $\frac{1}{13}$ which are slightly less than $\frac{1}{13}$. (**Hint**: $13 \times 7 = 91$, $13 \times 76 = 988$, $13 \times 769 = 9997$)

Think



Cut out an 8 by 13-cm rectangle and cut it into pieces as shown in the figure. Reassemble the pieces to form what seems to be a rectangle 5 by 21 cm.

1. What is the area of the 8 by 13-cm rectangle?
2. What is the area of the 5 by 21-cm rectangle?
3. Can you explain where the extra square centimeter is?

10-6 DIVIDING DECIMALS

Dividing numbers named by decimals is much like dividing whole numbers. Every division problem involving decimals can be replaced by a division problem that has a whole number divisor and yet has the same quotient as the original problem.

Discussion Exercises

1. The correct digits (not including zeros) are given for each quotient. Use estimation to find the correct quotient. Explain your work.

• Example:

$$8\overline{)58.48} \quad 731$$

→ 731 × 8 is approximately 5600, which is too large.
→ 73.1 × 8 is approximately 560, which is too large.
→ 7.31 × 8 is approximately 56, which is about right.
→ The quotient of 58.48 ÷ 8 is 7.31.

[a] $5\overline{)1.285}$ 257 [b] $34\overline{)241.4}$ 71 [c] $9\overline{)47.16}$ 524 [d] $34\overline{).9214}$ 271

2. Find an easy way to place the decimal point correctly in each problem in Exercise 1.

3. [a] Explain each step in the division example at the right.

[b] Check the quotient by multiplying .436 by 9.

Dividing tenths

$$9\overline{)3.924} \quad .4$$

Dividing hundredths

$$9\overline{)3.924} \quad .43$$

Dividing thousandths

$$9\overline{)3.924} \quad .436$$

4. Explain how the diagram shows that the quotient for $.9\overline{).3924}$ is the same as the quotient for $9\overline{)3.924}$.

$$.9\overline{).3924} \rightarrow \boxed{\frac{.3924}{.9} = \frac{.3924 \times 10}{.9 \times 10} = \frac{3.924}{9}} \rightarrow 9\overline{)3.924}$$

5. Explain how to find the missing number so that the two problems both have the same quotient.

[a] $.6\overline{)3.6} \rightarrow 6\overline{)\square}$ [b] $2.8\overline{)1.96} \rightarrow 28\overline{)\square}$ [c] $.19\overline{)8.749} \rightarrow 19\overline{)\square}$

6. You have found the quotient for $.9\overline{).3924}$ when you find the quotient for $9\overline{)3.924}$. Since the decimal point is easier to place when the divisor is a whole number, replace each problem with a problem that has the same answer and has a whole number divisor. Explain each example.

A Think: 10 × divisor. Think: 10 × dividend. Think about the division problem: $47\overline{)2.82}$. Complete the dividing.

$4.7\overline{).282}$ ⇒ $4.7\overline{).282}$ ⇒ $4.7\overline{).282}$

$$\begin{array}{r} .06 \\ 282 \end{array}$$

B Think: 100 × divisor. Think: 100 × dividend. Think about the division problem: $7\overline{)256.9}$. Complete the dividing.

$.07\overline{)2.569}$ ⇒ $.07\overline{)2.569}$ ⇒ $.07\overline{)2.569}$

$$37.7$$

Exercises

1. Find the quotients.

[a] $2\overline{)67.2}$ [c] $4\overline{)15.860}$ [e] $63\overline{).504}$ [g] $58\overline{)21.46}$

[b] $7\overline{)23.59}$ [d] $9\overline{)58.5}$ [f] $29\overline{)17.4}$ [h] $84\overline{)21.00}$

2. Give a division problem with a whole number divisor that has the same quotient as the problem given.

[a] $.4\overline{)29.6}$ [c] $.09\overline{)47.7}$ [e] $.008\overline{)3.568}$ [g] $.029\overline{).493}$

[b] $5.8\overline{).406}$ [d] $.25\overline{).625}$ [f] $3.45\overline{)24.15}$ [h] $61.8\overline{)5.562}$

3. Find the quotients.

[a] $.5\overline{).675}$ [e] $.06\overline{)3.708}$ [i] $.23\overline{).851}$ [m] $54.2\overline{)27.642}$

[b] $.7\overline{)305.9}$ [f] $.08\overline{)756.8}$ [j] $5.6\overline{)3416}$ [n] $.41\overline{)1.599}$

[c] $.2\overline{)538}$ [g] $.05\overline{).1385}$ [k] $.34\overline{)2.856}$ [o] $8.7\overline{)5.568}$

[d] $.6\overline{).5616}$ [h] $.07\overline{)15.218}$ [l] $.029\overline{)28.71}$ [p] $.643\overline{)263.63}$

4. Study the example. Then find each quotient, rounded to the nearest hundredth. **Annex zeros,** as in the example, when you need them.

[a] $8\overline{)3.0}$ [g] $.58\overline{).4683}$

[b] $7\overline{)32.54}$ [h] $.004\overline{)20.593}$

[c] $.9\overline{)42.345}$ [i] $9\overline{)4}$

[d] $.05\overline{)2.4730}$ [j] $.41\overline{).3457}$

[e] $6.1\overline{)31.76}$ [k] $2.9\overline{)1.672}$

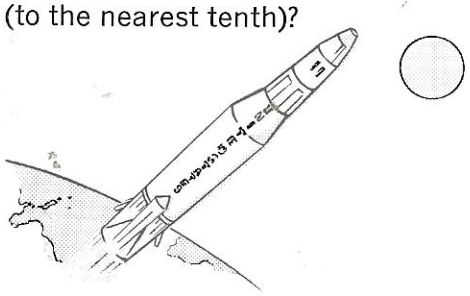
[f] $.39\overline{)2654}$ [l] $52\overline{).3854}$

$$\begin{array}{r} .725 \\ 2.3\overline{)1.6690} \\ \underline{1\ 61} \\ 59 \\ \underline{46} \\ 130 \\ \underline{115} \\ 15 \end{array}$$

We say: "the quotient, rounded to the nearest hundredth is .73."

10-7 SHORT STORIES—DECIMALS

1. Rocket to the moon: about 65 mi in 9 sec. How many miles per second (to the nearest tenth)?



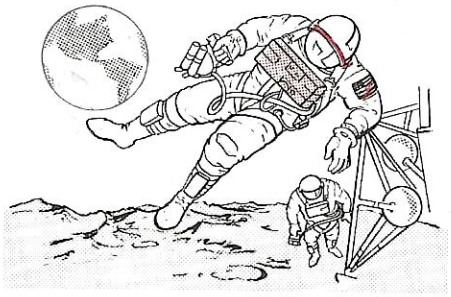
2. Rocket trip: 414 mi in 16.5 min. Averaged how many miles per minute (to nearest tenth)?

3. Diameter of Mars: about .53 times diameter of earth. Diameter of earth: about 7927 mi. What is diameter of Mars (to nearest hundred miles)?

4. Object weighs 6 times as much on earth as on moon. Boy: 94.5 lb; dog 4.3 lb. What is total "moon weight" of boy and dog (to nearest tenth)?

5. Rocket: To leave earth, 7 mi per sec. To leave Mars, .4 mi per sec less than half that for earth. How fast?

6. Weight of object on the moon: .16 times earth weight. Moon weight of astronaut: 24.8 lb. Earth weight?



7. Earth weight of cubic foot of water, 62.5 lb. Moon weight? (Exercise 6)

8. "Year" on Mars: 1.9 earth years. How many Mars years is 18.05 earth years?

9. Speed needed to leave moon: 1.5 mi per sec.

[a] How many miles per hour?

[b] Speed needed to leave Saturn 14.7 times as fast as for moon. How fast?

10. "Day" on moon: about 27.3 earth days. Day on Saturn 10.5 hr. Moon day is how many times greater (to nearest hundredth)?

11. Time needed to orbit earth: about 95.6 min. How many complete orbits in 8 hr?

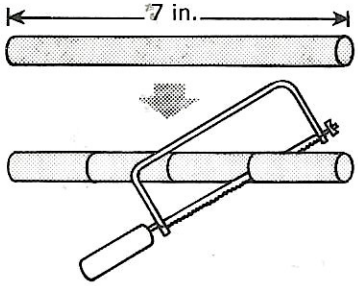
12. Weight of object on Mars .39 times its weight on earth. Astronaut's weight on Mars, 70.78 lb. Earth weight?

13. Steak, 2.5 lb
Potatoes, 10 lb
Flour, 5 lb
Grapes, 2.25 lb
Find the total "Mars weight" of these groceries. (See Exercise 12.)

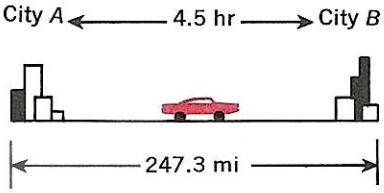
WORDLESS PROBLEMS

Each picture suggests a problem. Study the picture carefully. Then write and solve your own problem for the picture.

1.



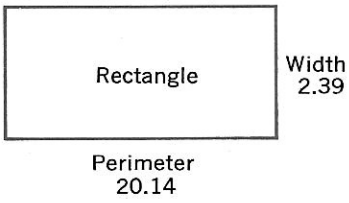
2.



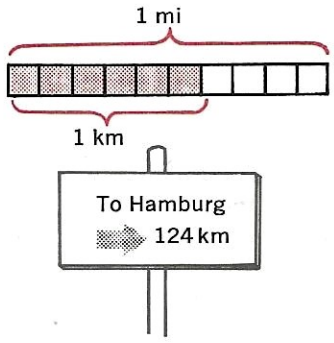
3.

Trip	Miles	Gallons used
1	562.3	41.2
2	467.5	34.6
3	227	16.2
4	389.3	28.7

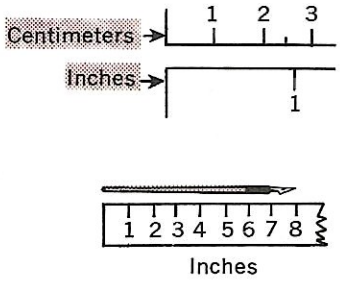
4.



5.



6.



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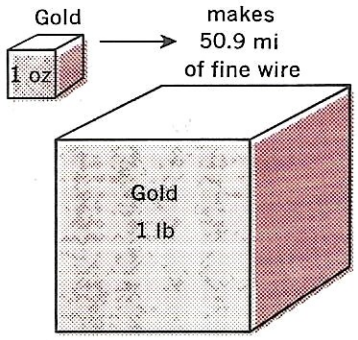
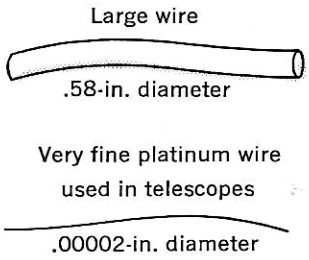
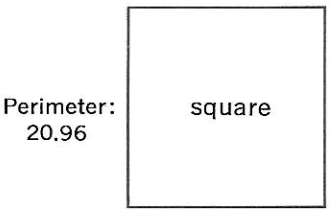


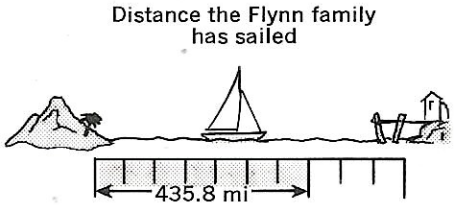
8.



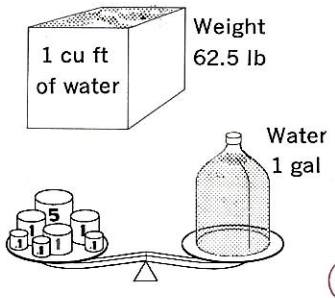
9.



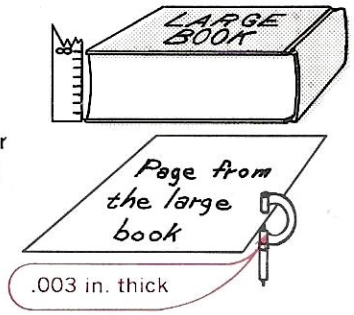
10.



11.



12.



Think

There is a shortcut for squaring numbers which are multiples of 5.

$$15 \cdot 15 = 10 \cdot 20 + n$$
$$25 \cdot 25 = 20 \cdot 30 + n$$
$$55 \cdot 55 = 50 \cdot 60 + n$$

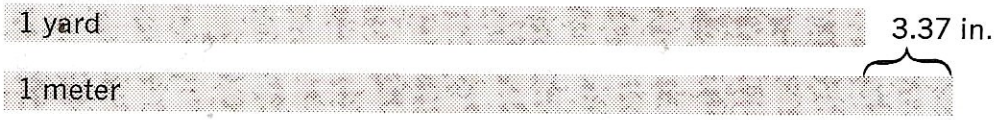
First find the number for n in each example above. Then, without using pencil and paper, determine the products below.

1. $65 \cdot 65$ **2.** $75 \cdot 75$ **3.** $85 \cdot 85$ **4.** $95 \cdot 95$

10-8 PROBLEMS ABOUT THE METRIC SYSTEM

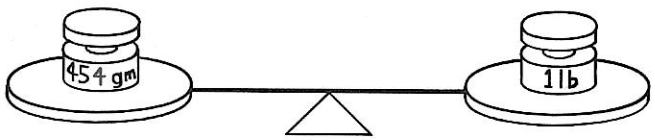
The chart compares the three basic units of the **metric system** (meter, gram, and liter) with the familiar units of the **English system** (yard, pound, and quart).

Unit of length → **Meter** (m) A meter is about 3.37 inches longer than a yard.



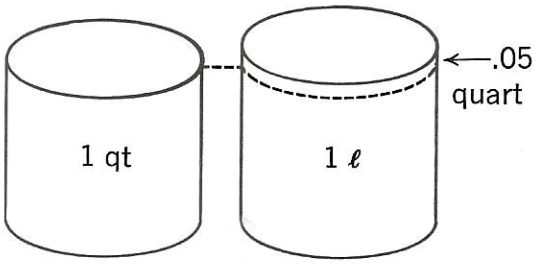
Unit of weight → **Gram** (gm)

An object that weighs 454 grams weighs about 1 pound.



Unit of capacity → **Liter** (ℓ)

A liter container holds $\frac{5}{100}$ of a quart more than a quart container.



One reason the metric system is so useful is that other units in the system are easily related to the basic three. For example, a **centi**meter is .01 meter, and a **kilo**gram is 1000 grams. You can understand the other units by studying the meaning of these prefixes.

milli- means **one thousandth**

centi- means **one hundredth**

deci- means **one tenth**

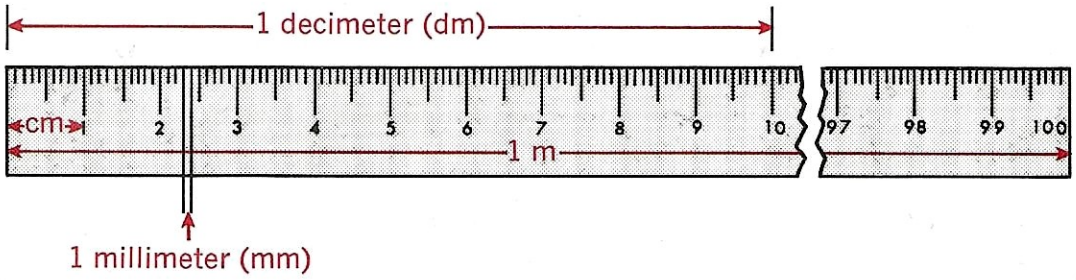
deca- means **ten**

hecto- means **one hundred**

kilo- means **one thousand**

Exercises

1. Study this diagram. Then give the missing numbers in each exercise.

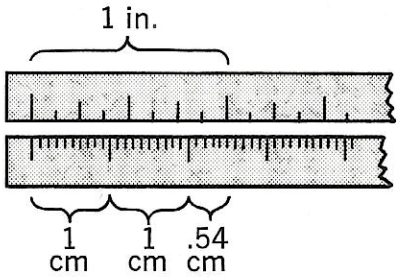


[a] 1 meter = ▒▒ dm
[b] 1 meter = ▒▒ cm
[c] 1 meter = ▒▒ mm
[d] 1 dm = ▒ m
[e] 1 cm = ▒ m
[f] 1 mm = ▒ m
[g] 1 cm = ▒▒ mm
[h] 1 dm = ▒▒ cm
[i] 1 mm = ▒ cm
[j] 1 cm = ▒ dm
[k] 8 dm = ▒ m
[l] 25 cm = ▒ m
[m] 59 cm = ▒ m
[n] 4 mm = ▒ m
[o] 29 mm = ▒ cm

2. Give the missing numbers.

[a] 1 kilometer = ▒ m
[b] 1 m = ▒ kilometer
[c] 1 hectometer = ▒ m
[d] 1 decimeter = ▒ m
[e] 1 m = ▒ millimeter
[f] 1 m = ▒ hectometer

3. One inch is a little more than 2.54 cm. Multiply 39.37 (the number of inches in 1 m) by 2.54 (the number of centimeters in 1 in.). By how much does your answer differ from 100?



4. The Olympic Games include a 100-m dash rather than a 100-yd dash. Would you be able to run the 100-m dash more or less quickly than the 100-yd dash?

5. [a] Show that 1 m is a little more than 3.28 ft.
[b] Show that 1 m is a little more than 1.09 yd.

6. A football field is 100 yd long. How many meters long is it?

7. [a] Using a decimal, express 742 in. as feet.
[b] Using a decimal, express 742 cm in meters.

8. [a] Using a decimal, express 23,735 ft as miles.
[b] Express 23,735 m in kilometers.

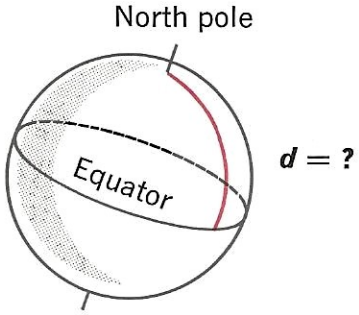
9. [a] Express 387 yd in inches.
[b] Express 387 m in centimeters.

10. How many millimeters are there in 1 in.? Is 1 mm less than $\frac{1}{16}$ in.? Less than $\frac{1}{32}$ in.?

11. [a] How many inches are there in 1 mi?
[b] How many centimeters in 1 km?

12. Approximately how many centimeters are there in 1 ft? In 1 yd?

13. Assume that the circumference of the earth is 24,000 mi and that 1 km is .6 mi. Compute the distance in meters from the north pole to the equator.



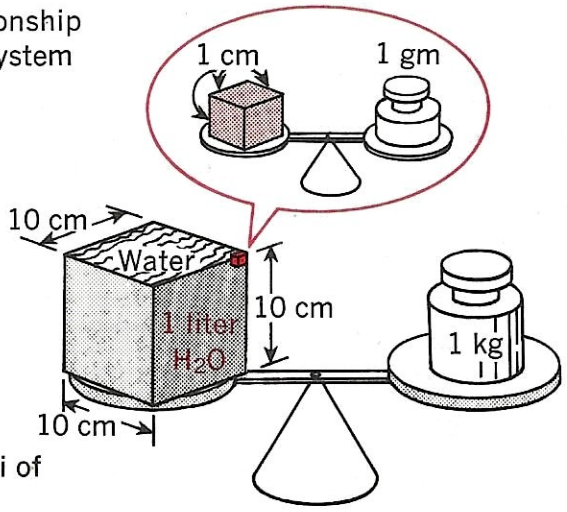
14. An object that weighs 1 kg weighs how many pounds?

15. Mr. Buzz put 5 ℓ of oil in his automobile. How many quarts of oil was this?

16. Diamonds are weighed in **carats**. A carat is .2 gm.
[a] How many 1-carat diamonds would it take to make up 1 gm?
[b] How many 1-carat diamonds would it take to make 1 lb?

17. A diamond crystal called the "great Cullinan crystal" was found in a mine in South Africa in 1905. In the rough, this crystal weighed 3106 carats. Express this weight in grams and then in ounces.

★**18.** The diagram shows an important relationship between the basic units of the metric system (meter, gram, and liter). Study it and give the missing numbers.



[a] 1 cu cm of water weighs ▒ gm.

[b] 1 ℓ has a volume of ▒ cu cm.

[c] 1 ℓ of water weighs ▒ gm.

[d] 10 cu cm = ▒ ℓ

[e] 1 ℓ has a volume ▒ of 1 cu m.

★**19.** Estimate the weight in grams of 1 cu mi of sea water.

Think

1. Suppose that the metric system, with the prefixes, **milli**, **centi**, **deci**, **kilo**, were used to measure time.

   [a] If your class period were a "deciday" long, would you like it better?

   [b] About how many "centidays" would it take to make an hour?

   [c] Which would be longer, a "milliday" or one minute?

   [d] What do you think might be meant by a "decimilliday"? A "centimilliday"? Find how many seconds each of these would be.

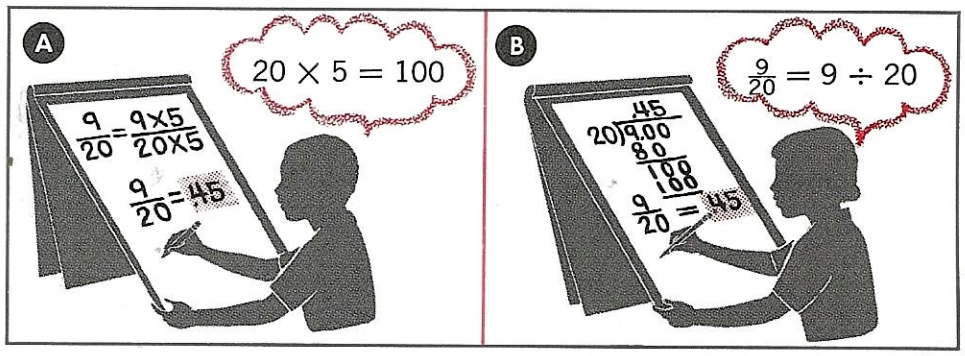
   [e] Assuming that the day starts at midnight, what time would be indicated by 0.375 of a day?

   [f] Draw a decimal clockface.

2. What might be meant by a "kilobuck"? A "centibuck"?

10-9 FROM FRACTIONS TO DECIMALS

You have been using both fractions and decimals as numerals for rational numbers. When you are given a decimal, it is easy to write a fraction that names the same number. When you are given a fraction, can you always write a decimal that names the same number? The examples show two ways to find a decimal for $\frac{9}{20}$.



Discussion Exercises

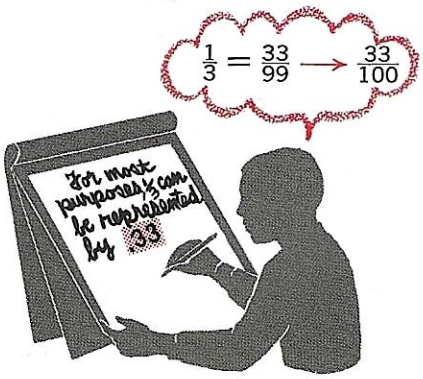
1. Explain how to use each of the methods shown above to find a decimal for $\frac{4}{5}$.

2. Explain how to use each of the methods shown above to find a decimal for $\frac{3}{8}$.

3. Which method do you think is easier?

4. Explain how to use each Method A and B to find the decimal for the fraction $\frac{3}{25}$.

5. This picture shows how Mike found a decimal for $\frac{1}{3}$. Do you think Mike has found a decimal that represents the same number as the fraction $\frac{1}{3}$? Explain your answer.



6. Can you use Method A or Method B shown above to find a decimal that represents the same number as the fraction $\frac{1}{3}$? Explain your answer.

Exercises

1. Give the number for n. Then give the correct decimal.

[a] $\frac{23}{50} = \frac{n}{100} = ?$ [b] $\frac{4}{5} = \frac{n}{10} = ?$ [c] $\frac{7}{25} = \frac{n}{100} = ?$

[d] $\frac{23}{200} = \frac{n}{1000} = ?$ [e] $\frac{9}{500} = \frac{n}{1000} = ?$ [f] $\frac{64}{250} = \frac{n}{1000} = ?$

[g] $\frac{3}{4} = \frac{n}{100} = ?$ [h] $\frac{17}{125} = \frac{n}{1000} = ?$ [i] $\frac{23}{80} = \frac{n}{10{,}000} = ?$

2. Find the quotients. In each problem, continue dividing until the remainder is zero. Then give the correct decimal.

[a] $8\overline{)5} \longrightarrow \frac{5}{8} = ?$ [b] $25\overline{)6} \longrightarrow \frac{6}{25} = ?$ [c] $2\overline{)27} \longrightarrow \frac{27}{2} = ?$

[d] $16\overline{)1} \longrightarrow \frac{1}{16} = ?$ [e] $16\overline{)9} \longrightarrow \frac{9}{16} = ?$ [f] $80\overline{)7} \longrightarrow \frac{7}{80} = ?$

3. Write a decimal for each number. Use any method you choose.

[a] $\frac{3}{5}$	[e] $\frac{7}{20}$	[i] $\frac{29}{40}$	[m] $\frac{7}{200}$	[q] $\frac{372}{500}$
[b] $\frac{1}{4}$	[f] $\frac{19}{25}$	[j] $\frac{43}{80}$	[n] $\frac{9}{400}$	[r] $\frac{98}{200}$
[c] $\frac{7}{8}$	[g] $\frac{27}{50}$	[k] $\frac{99}{150}$	[o] $\frac{569}{500}$	[s] $\frac{120}{150}$
[d] $\frac{11}{16}$	[h] $\frac{3}{2}$	[l] $\frac{147}{160}$	[p] $\frac{4}{25}$	★[t] $\frac{964}{1250}$

★**4.** Study the examples. Then write a decimal that represents a number equal to or approximately equal to the number given. The symbol $\approx$ means "is approximately."

(A) $\frac{37}{99} \approx \frac{37}{100} = .37$

(B) $\frac{56}{300} = \frac{56 \div 3}{300 \div 3} \approx \frac{19}{100} = .19$

(C) $\frac{43}{65} = \frac{4300}{6500} = \frac{4300 \div 65}{6500 \div 65} \approx \frac{66}{100} = .66$

[a] $\frac{17}{50}$	[f] $\frac{73}{999}$	[k] $\frac{17}{90}$	[p] $\frac{4672}{6000}$	[u] $\frac{79}{101}$
[b] $\frac{7}{99}$	[g] $\frac{63}{501}$	[l] $\frac{412}{2000}$	[q] $\frac{72}{13}$	[v] $\frac{38}{499}$
[c] $\frac{12}{200}$	[h] $\frac{41}{300}$	[m] $\frac{614}{1001}$	[r] $\frac{1}{13}$	[w] $\frac{840}{5000}$
[d] $\frac{23}{600}$	[i] $\frac{41}{700}$	[n] $\frac{46}{250}$	[s] $\frac{27}{75}$	[x] $\frac{63}{99}$
[e] $\frac{7}{49}$	[j] $\frac{11}{30}$	[o] $\frac{217}{3000}$	[t] $\frac{14}{37}$	[y] $\frac{19}{33}$

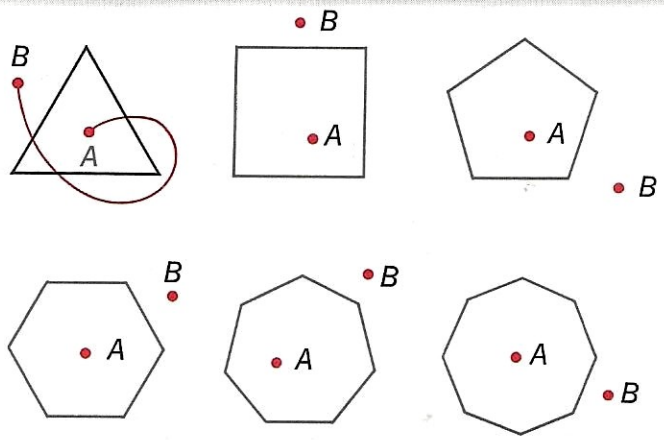
★**5.** In each exercise, solve the equation for n. Then find a fraction for x.

[a] $7 \times .42857 = n$
$.42857 \approx x$

[b] $19 \times .8421 = n$
$.8421 \approx x$

[c] $9 \times .22222 = n$
$.22222 \approx x$

Think



A **triangle**, a **square**, a **pentagon**, a **hexagon**, a **septagon**, and an **octagon** are pictured above.

Copy each figure and try to connect the **interior** point *A* to the **exterior** point *B* with a continuous curve that crosses each side of the polygon once and only once before reaching *B*. The first problem is completed as an example.

10-10 REPEATING DECIMALS

You can find a decimal for a rational number if a fraction having a denominator which is a power of ten can be found for the number. Also, this can be done if (in any fraction for the number) when the numerator is divided by the denominator the remainder is eventually zero.

Discussion Exercises

1. Find the whole number for n. Then write decimal names for $\frac{2}{5}$, $\frac{11}{25}$, and $\frac{3}{8}$. Explain how you found the number for n.

   [a] $\frac{2}{5} = \frac{n}{10}$ [b] $\frac{11}{25} = \frac{n}{100}$ [c] $\frac{3}{8} = \frac{n}{1000}$

2. Can you find a whole number for n? Explain your answer.

   [a] $\frac{1}{3} = \frac{n}{10}$ [b] $\frac{1}{3} = \frac{n}{100}$ [c] $\frac{1}{3} = \frac{n}{1000}$

3. Do you think $\frac{1}{3}$ can be represented by a fraction with a denominator that is a power of ten? Explain your answer.

4. Study this example. Will the remainder ever be zero?

$$3\overline{)1.0}\ .3,\ R\ 1 \qquad 3\overline{)1.0^{1}0}\ .33,\ R\ 1 \qquad 3\overline{)1.0\,0^{1}0}\ .333,\ R\ 1 \quad \ldots$$

Dividing tenths — Dividing hundredths — Dividing thousandths

5. Can $\frac{1}{3}$ be represented by the kind of decimals we have been studying? Explain your answer.

You have found that not all rational numbers can be represented by the kind of decimals we have been studying. We invent a new type of decimal to represent such numbers.

We write ordinary decimals to represent numbers like these.

$\frac{3}{10} = .3$

$\frac{7}{25} = .28$

$\frac{9}{16} = .5625$

We write **repeating decimals** to represent numbers like these.

$\frac{1}{3} = .33\ldots$

$\frac{5}{11} = .4545\ldots$

$\frac{1}{27} = .037037\ldots$

The three dots in $.33\ldots$ indicate that the threes repeat indefinitely. Also, they remind you how this decimal differs from .33, .333, .3333, and so on. In $.4545\ldots$, the dots indicate that the digits "45" repeat. In $.037037\ldots$, the dots indicate that the digits "037" repeat.

Exercises

1. Divide to find the repeating decimal that represents each rational number.

[a] $\frac{2}{3}$ [b] $\frac{1}{6}$ [c] $\frac{7}{11}$ [d] $\frac{1}{9}$ [e] $\frac{5}{27}$ [f] $\frac{5}{6}$ [g] $\frac{4}{11}$ [h] $\frac{2}{37}$ [i] $\frac{7}{15}$ [j] $\frac{5}{13}$ [k] $\frac{1}{7}$ [l] $\frac{5}{9}$ [m] $\frac{5}{12}$ ★[n] $\frac{5}{17}$

2. The examples show another way that is often used to write the repeating decimal for a rational number.

(A) $\frac{1}{3} = .33\ldots = .\overline{3}$

(B) $\frac{5}{11} = .4545\ldots = .\overline{45}$

(C) $\frac{1}{27} = .037037\ldots = .\overline{037}$

The bar written over the group of digits that repeats is a short way of showing a repeating decimal.

Use a bar to write the repeating decimals you found in Exercises 1**a** through **n**.

3. Round each repeating decimal you found in Exercise 1 to the nearest hundredth.

★**4.** Find the repeating decimal that represents $\frac{1}{81}$. What interesting fact do you notice about this decimal?

★**5.** Given a fraction that represents a rational number, you can divide to find a repeating decimal for the number. When given a repeating decimal that represents a rational number, can you find a fraction for the number? The following exercises will help you see how to do this. Copy each exercise. Give a repeating decimal for each color screen and a fraction for each gray screen.

[a] $\frac{1}{9} =$ ▒▒

[b] $\frac{1}{99} =$ ▒▒

[c] $\frac{1}{999} =$ ▒▒

[d] ▒ = .111 . . .
+▒ = +.111 . . .
▒▒

(**Hint:** Add the repeating decimals from left to right.)

[e] ▒ = .222 . . .
+▒ = +.111 . . .
▒▒

[f] ▒ = .0101 . . .
+▒ = +.0101 . . .
▒▒

[g] ▒ = .0202 . . .
+▒ = +.0101 . . .
▒▒

[h] ▒ = .1111 . . .
+▒ = +.0101 . . .
▒▒

[i] ▒ = .3333 . . .
+▒ = +.0202 . . .
▒▒

[j] ▒ = .001001 . . .
+▒ = +.111111 . . .
▒▒

[k] ▒ = .002002 . . .
+▒ = +.333333 . . .
▒▒

★**6.** Write a fraction that represents the same number as $.\overline{347}$.

Think

In the magic square to the right, the sum of each row, column, diagonal, and the four corners is the same.

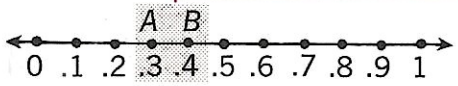
1. To complete the magic square, use each of the numbers 1 through 16 just once. To help you, some of the numbers are already given.
2. What is the sum?

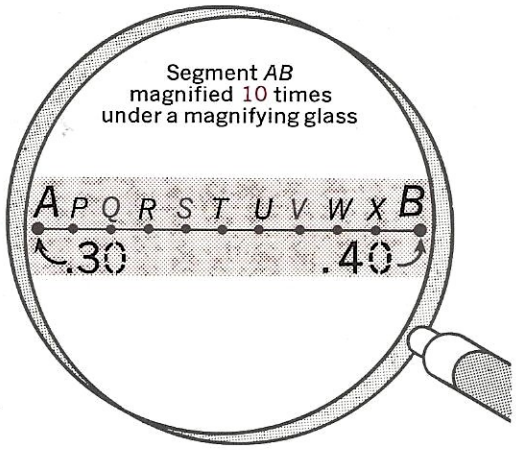
1			15
	13		6
7		16	9
14	11		

10-11 INEQUALITIES AND THE NUMBER LINE

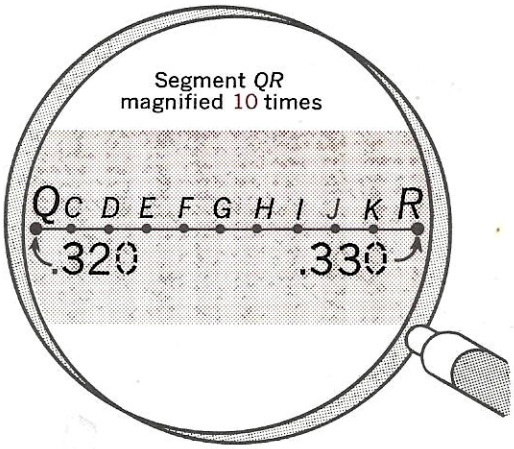
Discussion Exercises

It is interesting to think about matching decimals with points on the number line. Study the pictures and answer the questions about graphing the number .327 on the number line.





1. The point for .327 is between .**300** and .**400**. Is it closer to .3 or to .4? Explain your answer.



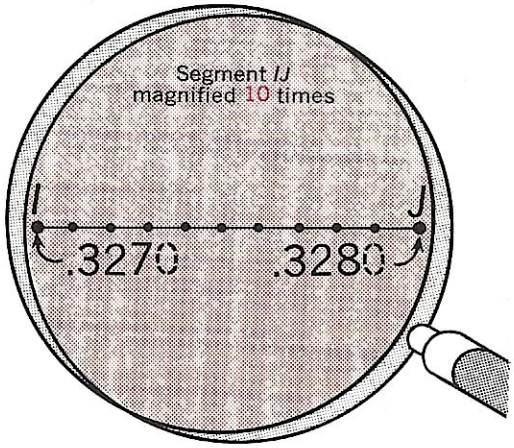
2. [a] Using the magnified segment *AB*, give the decimals that go with each point *P* through *X*.

[b] Between what two points is the point for .327?

3. [a] Using the magnified segment *QR*, give the decimals that go with each point *C* through *K*.

[b] Which point is the graph of .327?

[c] Between what two points is the graph of .3274?



4. Write nine decimals that represent numbers between .327 and .328.

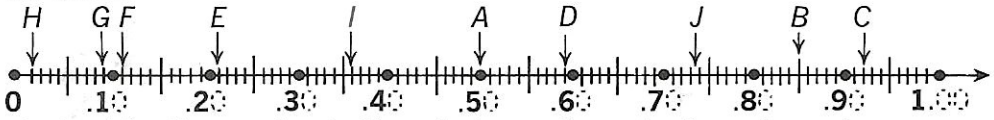
★**5.** If the segment from 0 to 1 is the unit, give the length of

[a] $\overline{AB}$ [b] $\overline{QR}$ [c] $\overline{IJ}$

Explain your reasoning.

Exercises

1. Starting with point *A*, give the decimals that go with the point below the arrow.



2. Give a decimal for the number halfway between the pair of numbers given.

Example: .6 → *n* → .7; *n* = .65

[a] .40 → *n* → .50
[b] 7.40 → *n* → 7.50
[c] .10 → *n* → .20
[d] 0.00 → *n* → .10
[e] .320 → *n* → .330
[f] .970 → *n* → .980
[g] 19.860 → *n* → 19.870
[h] .6340 → *n* → .6350
[i] .9 → *n* → 1.0
[j] .2 → *n* → .3
[k] .74 → *n* → .75
[l] .826 → *n* → .827

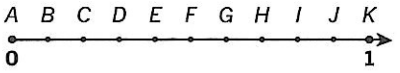
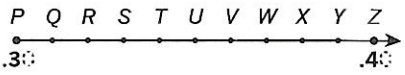
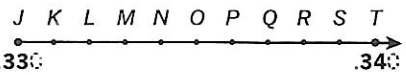
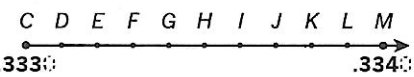
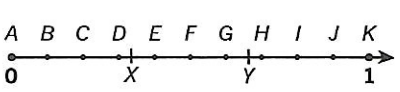
3. Give decimals for nine numbers between each pair of numbers given. Thinking about the number line may help you.

[a] .3, .4 [b] .7, .8 [c] .9, 1 [d] .53, .54 [e] .372, .373

4. [a] Give the decimal for the number halfway between .87659462 and .87659463.

[b] Do you think there is a number between **any two** numbers represented by decimals? Is there more than one?

5. In each row of the table, give the point on the number line that goes with the number indicated.

Fraction	Decimal	Point on the number line
$\frac{3}{10}$	.3	
$\frac{33}{100}$	.33	
$\frac{333}{1000}$	.333	
$\frac{3333}{1000}$	.3333	
⋮	⋮	
$\frac{1}{3}$	$.\overline{3}$	

6. Think about the repeating decimal for each fraction. Write the correct symbol ($<$, $>$, or $=$) for each ◯.

[a] $.3 \bigcirc \frac{1}{3}$ [e] $.1 \bigcirc \frac{1}{7}$ [i] $.6 \bigcirc \frac{2}{3}$ [m] $.1 \bigcirc \frac{1}{9}$

[b] $.4 \bigcirc \frac{1}{3}$ [f] $\frac{1}{7} \bigcirc .2$ [j] $\frac{2}{3} \bigcirc .7$ [n] $.12 \bigcirc \frac{1}{9}$

[c] $.333 \bigcirc \frac{1}{3}$ [g] $.14 \bigcirc \frac{1}{7}$ [k] $.6666 \bigcirc \frac{2}{3}$ [o] $.111 \bigcirc \frac{1}{9}$

[d] $.334 \bigcirc \frac{1}{3}$ [h] $\frac{1}{7} \bigcirc .15$ [l] $\frac{2}{3} \bigcirc .6667$ [p] $.112 \bigcirc \frac{1}{9}$

In Exercise 6, you have seen that → $.3 < \frac{1}{3}$ and $\frac{1}{3} < .4$.
We can write these two inequalities more simply as $.3 < \frac{1}{3} < .4$.
When you see $.3 < \frac{1}{3} < .4$, you might think, "$\frac{1}{3}$ is between .3 and .4."

In Exercises 7 through 9, we are "closing in" on a rational number. Give the decimal for ℓ (lower bound) and u (upper bound) in each exercise.

★7. Ⓐ Nearest tenth: $\ell < \frac{1}{3} < u$ (Answer: $\ell = .3$, $u = .4$)

Ⓑ Nearest hundredth: $\ell < \frac{1}{3} < u$ (Answer: $\ell = .33$, $u = .34$)

[a] Nearest thousandth: $\ell < \frac{1}{3} < u$

[b] Nearest ten thousandth: $\ell < \frac{1}{3} < u$

[c] Nearest hundred thousandth: $\ell < \frac{1}{3} < u$

★8.

[a] Nearest tenth: $\ell < \frac{7}{11} < u$

[b] Nearest hundredth: $\ell < \frac{7}{11} < u$

[c] Nearest thousandth: $\ell < \frac{7}{11} < u$

[d] Nearest ten thousandth: $\ell < \frac{7}{11} < u$

[e] Nearest hundred thousandth: $\ell < \frac{7}{11} < u$

★9.

[a] Nearest tenth: $\ell < \frac{1}{7} < u$

[b] Nearest hundredth: $\ell < \frac{1}{7} < u$

[c] Nearest thousandth: $\ell < \frac{1}{7} < u$

[d] Nearest ten thousandth: $\ell < \frac{1}{7} < u$

[e] Nearest hundred thousandth: $\ell < \frac{1}{7} < u$

★**10.** Since $\frac{1}{3} - \frac{3}{10} = \frac{1}{30}$, we say the difference between $\frac{1}{3}$ and .3 is $\frac{1}{30}$.

[a] What is the difference between $\frac{1}{3}$ and .33?

[b] What is the difference between $\frac{1}{3}$ and .333?

[c] What is the difference between $\frac{1}{3}$ and .3333?

[d] Write a decimal so that the difference between $\frac{1}{3}$ and this decimal is $\frac{1}{3{,}000{,}000}$.

[e] Suppose someone chooses a very small number. Can you always write a decimal so that the difference between $\frac{1}{3}$ and your decimal is less than the very small number?

10-12 SCIENTIFIC NOTATION—DECIMALS

A number other than a power of ten is represented using **scientific notation** when it is written as the product of a **number between 1 and 10 and a power of ten.** In the table, each number in Column A is represented in Column B using scientific notation.

(A) Number		(B) Number between 1 and 10		Power of ten
864	=	8.64	×	10^2
3785.3	=	3.7853	×	10^3
57	=	5.7	×	10^1
89,600	=	8.96	×	10^4

For a power of ten, such as 1,000,000, we might write 1×10^6.

Exercises

1. Find the number for ***n*** in each row of the table.

2. Represent the numbers in scientific notation.

[a] 700
[b] 900
[c] 8000
[d] 70,000
[e] 600,000
[f] 100,000

Number		Number between 1 and 10		Power of ten
[a] 37	=	***n***	×	10^1
[b] 467	=	4.67	×	***n***
[c] 6843	=	***n***	×	10^3
[d] ***n***	=	4.756	×	10^3
[e] 4800	=	***n***	×	10^3
[f] 96,000	=	9.6	×	***n***
[g] ***n***	=	5.83726	×	10^2
[h] 9463.21	=	***n***	×	10^3
[i] 5,800,000	=	5.8	×	***n***
[j] ***n***	=	3.7	×	10^6

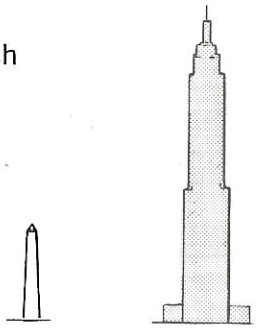
3. Which of the products are written in scientific notation?

[a] 4.32×10^3 [d] 1.14×10^4 [g] 0.01×10^4

[b] 53×10^2 [e] 4.3×10^2 [h] 6.537×10^6

[c] 0.0034×10^5 [f] 10×10^4 [i] 34×10^3

4. Convert to scientific notation those numbers in Exercise 3 which are not written in this notation.



Washington Monument

Empire State Building

5. Represent the numbers in scientific notation. Each number is a distance measured in feet.

[a] Height of Washington Monument: 555

[b] Height of Empire State Building: 1472

[c] Height of Mt. Everest: 29,000

[d] Length of Missouri-Mississippi River: 21,000,000

[e] Distance from Mars to satellite Phobos: 30,800,000

[f] Total length of railroad track in the United States in 1956: 1,153,000,000

[g] Diameter of a giant star: 9,100,000,000,000

[h] Distance light travels in one year: 31,000,000,000,000,000

[i] Distance to nearest star beyond our sun: 132,000,000,000,000,000

[j] Diameter of the Milky Way: 3,000,000,000,000,000,000,000

[k] Distance to extragalactic nebulae: 31,000,000,000,000,000,000,000

Mt. Everest

6. Represent the numbers in scientific notation. Each number is a weight given in pounds.

[a] Average man: 160

[b] Large elephant seal: 8000

[c] World's heaviest bell: 432,000

[d] Atmosphere of the earth: 11,400,000,000,000,000,000

[e] Sun: 4,000,000,000,000,000,000,000,000,000,000

[f] Milky Way: $\underbrace{600{,}000, \ldots 000}_{41 \text{ zeros}}$

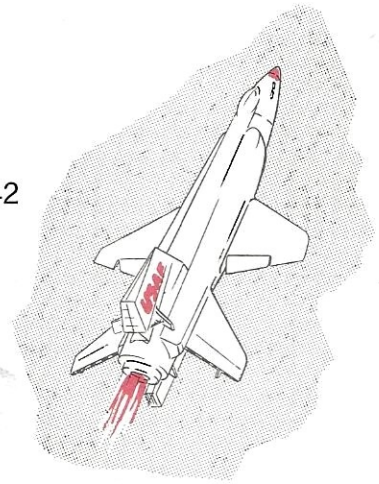
[g] Universe (according to Einstein): $\underbrace{40{,}000, \ldots, 000}_{52 \text{ zeros}}$

7. Represent the numbers in scientific notation. Each number is a period of time given in seconds.

[a] Hour: 3600
[b] Day: 86,400
[c] Month: 2,550,000
[d] Solar year: 31,556,900
[e] Average life of man: 2,000,000,000
[f] Revolution of Pluto about the sun: 7,820,000,000
[g] Half-life of radium 226: 52,400,000,000
[h] One rotation of the Milky Way: 6,000,000,000,000,000
[i] Estimated age of the sun: 100,000,000,000,000,000
[j] Half-life of uranium 238: 142,000,000,000,000,000

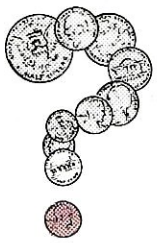
8. Represent the numbers in scientific notation. Each number is a velocity given in miles per hour.

[a] Sprinter: 21.7
[b] Race horse: 42.3
[c] Eagle: 120
[d] Propeller driven plane: 500
[e] Sound traveling through air at standard conditions: 742
[f] A rocket-engine plane: 4000
[g] Manned satellite: 17,600
[h] Escape velocity of rocket from earth: 25,000
[i] Earth in orbit around the sun: 66,600
[j] Hydrogen electron around its nucleus: 4,700,000
[k] Light: 669,600,000



Think

What is the greatest amount of money you can have (using pennies, nickels, dimes, quarters, and half dollars) and still not be able to give someone change for a dollar?



CHAPTER REVIEW

Key Concepts

1. A **decimal** is a numeral used to represent certain rational numbers. Any rational number that can be named by a fraction having a denominator a power of ten can be named by a decimal. Decimals are often easier to use and take up less space than fractions. The concept of **place value** is extended to give meaning to decimals.
2. Adding, subtracting, multiplying, and dividing using decimals is much like performing these same operations with whole numbers. Your work with fractions will also help you understand decimal operations.
3. It is useful to be able to find **approximations** to certain numbers represented by decimals. These approximations help you **estimate** products and quotients in multiplication and division problems.
4. The metric system is based on the decimal system. If you understand decimals clearly, you can better understand the relationships between units in the metric system.
5. Not every rational number can be represented by ordinary decimals such as .5, .37, and 2.56. **Repeating decimals** are used to represent some rational numbers, such as $\frac{1}{3} = .33\ldots = .\overline{3}$
6. Decimal notation is useful in graphing rational numbers on the number line. Using decimals, you can easily show that between any two rational numbers there is another rational number.
7. Scientific notation is useful as a simple way to represent large numbers. A number is represented in scientific notation when it is expressed as the product of a number between 1 and 10 and a power of 10.
   Example: $328{,}000{,}000 = 3.28 \times 10^8$

Vocabulary List

approximation	decimal point	liter	power
carat	estimation	meter	repeating decimal
decimal	gram	metric system	scientific notation

Review Exercises

1. In the decimal **.347516**, which digit tells how many
   [a] hundredths
   [b] ten thousandths
   [c] tenths
   [d] millionths
   [e] thousandths
   [f] hundred thousandths
2. Write the decimal for
   $7 \cdot 100 + 5 \cdot 10 + 3 \cdot 1 + 2 \cdot \frac{1}{10} + 8 \cdot \frac{1}{100}$
3. Write 26.3745, using expanded notation (exponent form).
4. Write a decimal for $5 + \frac{376}{1000}$.
5. Write a fraction for 72.0936.
6. Compute the following.
   [a] $3.7 + 59.642 + 463.24 + .803$
   [b] $597.7 - 136.9$
   [c] $\begin{array}{r} .59 \\ \times 6.8 \\ \hline \end{array}$
   [d] $\begin{array}{r} 42.8 \\ \times .006 \\ \hline \end{array}$
   [e] $8\overline{).2776}$
   [f] $.23\overline{)1.978}$
   [g] $5\overline{)16}$
   [h] $16\overline{)5}$
7. Find the products. Write only the answer.
   [a] 10×5.736
   [b] $100 \times .856$
   [c] 1000×5.42
   [d] $.1 \times 87.54$
   [e] $.01 \times 956$
   [f] $.001 \times 87.6$

8.

	Miles traveled	Gallons used	Cents per gallon
Sat.	297.6	15.2	36.9
Sun.	164.5	8.7	36.9

[a] Find the total number of miles traveled.
[b] Find the total number of gallons used.
[c] What was the total cost of the gasoline used on the weekend?
[d] How many miles per gallon (to the nearest tenth) did the car average on Saturday?
[e] How many miles per gallon (to the nearest tenth) did the car average on Sunday?

9. Round **57.5649** to the
[a] nearest whole number
[b] nearest tenth
[c] nearest hundredth
[d] nearest thousandth

10. [a] Estimate the product 7.96×49.84.
[b] Find the product in Part **a**. How far off was your estimate?

11. If you are in Europe and notice a sign stating that there are 59 kilometers to Geneva, do you have more or less than 59 miles to travel?

12. [a] Write a decimal for $\frac{4}{5}$.
[b] Write a decimal for $\frac{9}{32}$.
[c] Round the decimal you found in Part **b** to the nearest hundredth.

13. [a] Represent $\frac{9}{11}$, using a repeating decimal.
[b] Round the decimal in Part **a** to the nearest hundredth.

14. Give the decimal for the number halfway between .586 and .587

15. Write the correct symbol ($<$, $>$, or $=$) for each ◍.
[a] 53.099 ◍ 53.10
[b] .9 ◍ .900
[c] $\frac{2}{3}$ ◍ .667
[d] $\frac{7}{13}$ ◍ .53

16. Express the distance from the earth to the sun (93,000,000 mi) in scientific notation.

CHAPTER SUPPLEMENT

Reading List

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Activities

A. A **micrometer** is an instrument used for measuring small dimensions. Find a micrometer and see if you can figure out how to use it. Then use it to measure

- the thickness of a sheet of notebook paper
- the thickness of a pencil
- the thickness of a hair

B. Figure out how decimals are used in bases other than ten and be able to explain their use to your class. For example,

$$3.2_{(6)} = 3\tfrac{1}{3}_{(10)}$$
$$1.23_{(5)} = 1.52_{(10)}$$

A = 6
B = 1

C. Study a barometer and be able to explain the use and meaning of the decimals on it.

D. Decimals are used on cash registers, on spark plugs, and on automobile odometers. Find other uses for decimals and be ready to explain them to the class.

E. Most rulers show the inch divided into halves, fourths, eighths, and sixteenths. Using a heavy piece of cardboard, make your own ruler by dividing the inch into tenths. What would be some advantages of a ruler with tenths marked on it rather than eighths and sixteenths?

"*Alice was puzzled.* "*In* our *country,*" *she remarked,* "*there's only one day at a time.*"

The Red Queen said, "*That's a poor thin way of doing things. Now* here, *we mostly have days and nights two or three at a time, and sometimes in the winter we take as many as five nights together—for warmth, you know.*"

"*Are five nights warmer than one night, then?*" *Alice ventured to ask.*

"*Five times as warm, of course.*"

"*But they should be five times as* cold, *by the same rule ——*"

"*Just so!*" *cried the Red Queen.* "*Five times as warm,* and *five times as cold—just as I'm five times as rich as you are,* and *five times as clever!*"

11

Ratio, Proportion, and Percent

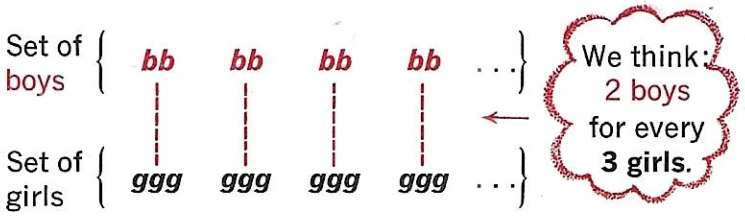
11-1 COMPARING SETS

We use a whole number to tell how many objects are in one set.

(A) We see: one set. { ♦ ♦ ♦ ♦ ♦ } ← We think: 5

We use the **language of ratio** to compare a first set with a second set.

(B) We see: two sets.

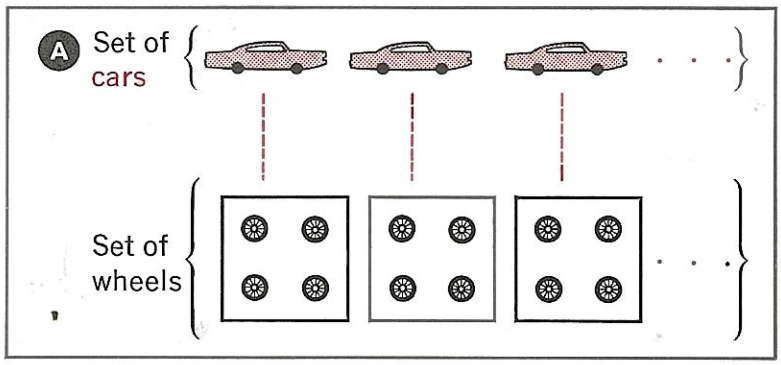


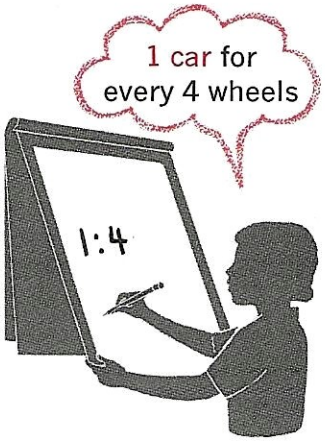
Using the language of rational numbers, the grouping shows us that there are $\frac{2}{3}$ as many boys as girls. In the special language of ratio, we say, "the **ratio** of boys to girls is 2 to 3," and we write, 2:3.

The language of ratio helps us to visualize relationships between sets and thus to solve special types of problems. Ratio language is a new language for talking and thinking about rational numbers.

Discussion Exercises

Questions 1 through 3 refer to Display A.





1. How can the sets be regrouped to show that there are
[a] 2 cars for every 8 wheels?
[b] 3 cars for every 12 wheels?
[c] 6 cars for every 24 wheels?

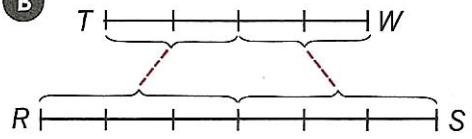
2. The fraction $\frac{1}{4}$ compares the number of cars with the number of wheels. What fraction compares the number of wheels with the number of cars?

3. Use the language of ratio to compare the number of cars with the number of wheels.

4. Look at Display B.
[a] What fraction compares $\overline{TW}$ with the unit-segment $\overline{RS}$?
[b] How would you say this, using a ratio instead of a fraction?

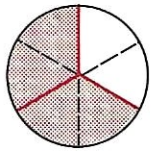
5. Look at Display C.
[a] What fraction tells us how much of the pie is left?
[b] Compare the part of pie left with the whole pie, using the language of ratio.

B



$\overline{RS}$ is the unit. Both $\overline{TW}$ and $\overline{RS}$ are divided into congruent segments.

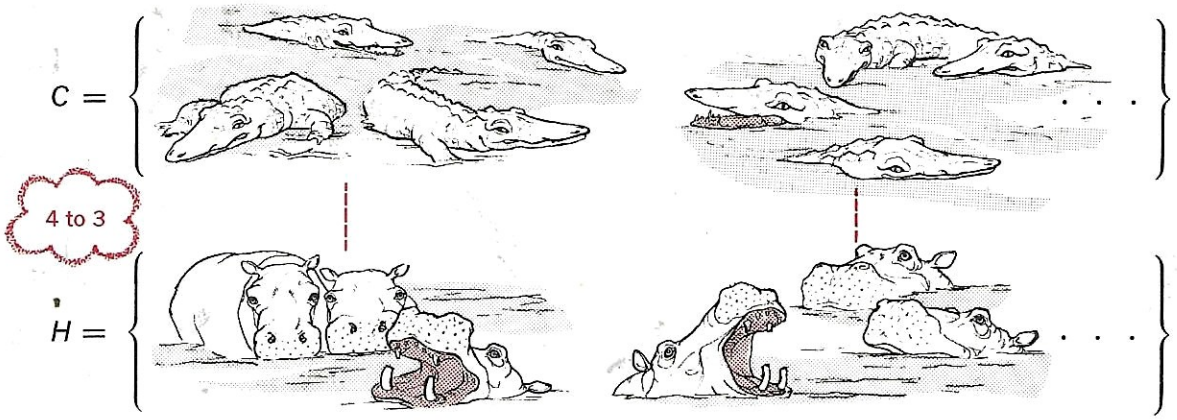
C



1 of 3 pieces of pie has been eaten.

Exercises

1. Think of a set C of crocodiles and a set H of hippopotamuses in the Nile river.



The ratio of crocodiles to hippos is 4 to 3.

[a] How many crocodiles are there if there are 30 hippos?

[b] How many crocodiles are there for every 6 hippos? For every 9 hippos? For every 300 hippos?

[c] Write three ratio symbols of the form 4:3 comparing the number of crocodiles with the number of hippos. For example, 8:6 compares crocodiles with hippos.

★[d] If there are 12 more crocodiles than hippos, how many crocodiles are there?

★[e] If there is a total of 140 crocodiles and hippos, how many hippos are there?

2. The symbol 5:3 compares the population of city A with the population of city B.

[a] To compare the two populations, sketch a figure like that in Exercise 1.

[b] Write three other $a:b$ symbols, such as 5:3 and 10:6, comparing A with B.

[c] Give three ratio symbols comparing B with A.

★[d] If there are 10,000 persons in A, how many are there in B?

★[e] If there are x persons in A and y in B, which statement is true?

(1) $x = \frac{5}{3} \cdot y$ (2) $y = \frac{5}{3} \cdot x$

★[f] If a total of 120,000 persons live in the two cities, how many live in each city?

3. The diagram compares the number of apple trees in an orchard with the number of peach trees.

[a] How many peach trees are there for every 2 apple trees? For every 4 apple trees? For every 10 apple trees?

[b] Give two ratio symbols comparing apple with peach trees.

[c] Give two ratio symbols comparing peach with apple trees.

[d] The ratio of apple trees to the total number of peach and apple trees is 2 : 7. What is the ratio of peach trees to all trees?

★[e] Suppose there are 12 more peach than apple trees. How many trees are in the orchard?

4. In a group of children, there are 7 girls for every 5 boys. Determine x if we say that

[a] there are 14 girls for every x boys

[b] there are 35 girls for every x boys

[c] there are x girls for every 15 boys

[d] the ratio of girls to boys is x to 50

[e] the ratio of girls to boys is 42 to x

[f] the ratio of boys to girls is x to 119

5. The ratio of set X to set Y is 3 : 4.

[a] Which set has the more elements?

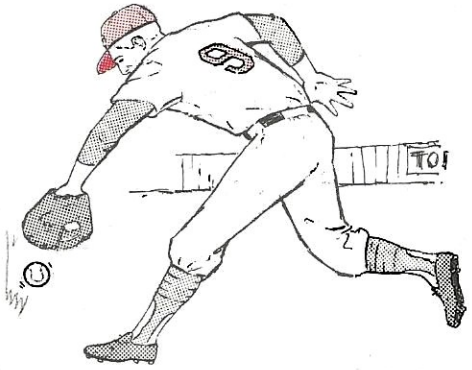
[b] If there are five more objects in one set than in the other, how many are in each set?

6. A ballplayer's ratio of number of hits to number of times at bat is 3 to 14.

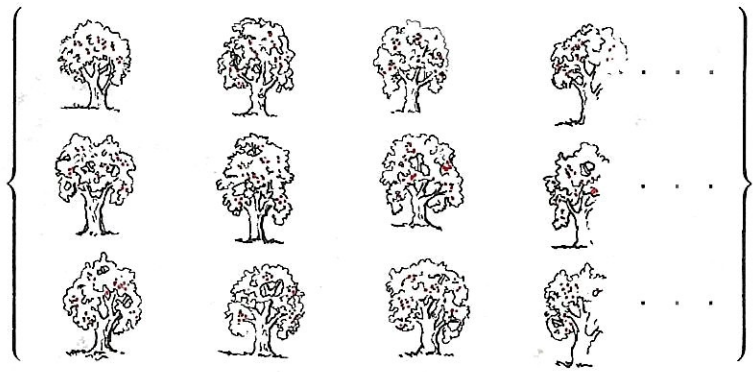
[a] Give two different symbols for the ratio which compares his number of hits with his number of times at bat.

[b] If the ballplayer has 24 hits, how many times has he been at bat?

[c] If the player has been at bat 210 times, how many hits has he?



7. In an orchard are 3 rows of cherry trees and 2 of plum trees. There are the same number of trees in each row.

$C =$ 

$P =$

[a] Give two ratios comparing cherry with plum trees.

[b] Give two ratios comparing plum with cherry trees.

[c] If there are 30 more cherry than plum trees, how many trees are in the orchard?

★[d] If 100 more plum trees were planted, there would be twice as many plum as cherry trees. How many cherry trees are there?

Think

The ratio of boys to girls in a group of children was 3 to 5. Then 24 girls left the group and 24 more boys joined it. The ratio of boys to girls became 5 to 3. How many boys and girls were in the original group?

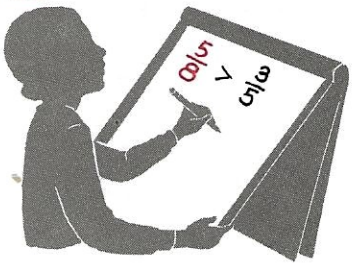
11-2 COMPARING RATIOS

We can decide which of two teams, Panthers and Wolverines, has the better record by comparing the ratio 3 to 5 with the ratio 5 to 8.

Team	Won	Games played	Ratio
Panthers	12	20	12:20 or 3:5
Wolverines	10	16	10:16 or 5:8

Since we know how to work with rational numbers, we think: "The Panthers have won **three fifths** of their games and the Wolverines **five eighths**."

Since $\frac{5}{8} > \frac{3}{5}$, the Wolverines have a better record than the Panthers.



Since $\frac{5}{8} > \frac{3}{5}$, we say that the ratio 5 to 8 is greater than the ratio 3 to 5 ($5:8 > 3:5$).

Also, we say that $4:8 = 1:2$, because $\frac{4}{8} = \frac{1}{2}$. A statement that two ratios are equal is called a **proportion**. In each proportion, we call two of the numbers **means** and the other two **extremes**.

Means (middle numbers)

$$3:6 = 4:8 \qquad 3 \cdot 8 = 6 \cdot 4$$

Extremes (end numbers)

Two ratios are equal if the product of the means is equal to the product of the extremes.

Corresponding to the **lowest-terms** language for fractions, we speak of writing ratios in **simplest terms**. For example, in simplest terms, we express the ratio of 15 to 10 as **3 to 2**.

Exercises

1. Express each ratio in simplest terms.
[a] 4 to 8 [b] 8 to 4 [c] 25:15 [d] 15:25 [e] 36:20

2. For each proportion, name the means. Verify that the product of the means is equal to the product of the extremes.
[a] 2:3 = 4:6 [b] 1:2 = 12:24 [c] 5 to 3 = 15 to 9 [d] 10 to 8 = 5 to 4

3. Copy and give the correct symbol (>, <, or =) for each ●.
[a] 3 to 2 ● 2 to 3 [c] 6 to 5 ● 1 to 1 [e] 15:9 ● 20:12 [g] 3:5 ● 55:100
[b] 3 to 5 ● 2 to 3 [d] 11 to 12 ● 6 to 5 [f] 1:2 ● 50:100 [h] 76:100 ● 3:4

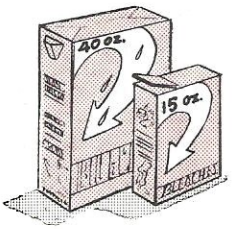
4. In a group, the ratio of men to children is 3:5, and the ratio of women to children is 5:8. Compare these ratios in order to decide whether there are more men or women in the group.

5. One ballplayer has a ratio of hits to times at bat of 5 to 16. Another ballplayer has a ratio of 3 to 10. Compare these ratios in order to decide which player has the better batting average.

6. One team has lost 7 of 18 games. Another team has lost 5 of 12 games.
[a] Which of the two ratios is greater, 7:18 or 5:12?
[b] Which team has the better record?

7. A 12-oz loaf of bread sells for 20 cents. A 16-oz loaf sells for 25 cents.
[a] Which of the two ratios is greater, 12 to 20 or 16 to 25?
[b] If there is no difference in the quality of the bread, which is the better buy?

8. A box of detergent marked to sell for 24 cents holds 15 oz. Another box of the same brand marked 65 cents contains 40 oz.
[a] Which of the two ratios is greater, 24 to 15 or 65 to 40?
[b] Which box is the better buy?



9. Car *A* travels 240 mi on 15 gal of gasoline. Under similar conditions, car *B* travels 200 mi on 12 gal.
[a] Write ratios that represent miles per gallon for each car.
[b] Which of these ratios is the larger?
[c] Which car uses gas more economically?

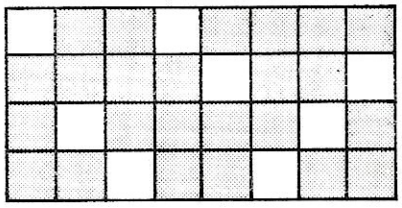
10. The ratio of a grasshopper's length to the distance he can jump is about 1 to 20. For a cricket the ratio is about 1 to 24.

[a] Which is the larger ratio?

[b] If the grasshopper and the cricket are the same size, which can jump farther?

11. Arrange the following ratios in order of size from the smallest to the largest: 4:5, 3:4, 8:7, and 1:1.

★ **12.** Compare the number of shaded squares in the rectangle with the total number of squares.



[a] Looking at the columns only, what is the natural way to describe the ratio of shaded squares to all squares?

[b] Looking at the horizontal rows only, what is the natural way to describe the ratio of shaded squares to all squares?

[c] Explain how this figure suggests the equality of the two ratios 3 to 4 and 6 to 8.

★ **13.** There are 7 apple for every 10 peach trees and 5 peach for every 3 cherry trees.

[a] Are there more apple or cherry trees?

[b] What is the ratio of apple to cherry trees?

★ **14.** The ratio of $\overline{AB}$ to $\overline{CD}$ is 18 to 8. The ratio of $\overline{CD}$ to $\overline{EF}$ is 2 to 5.

[a] Which is the longer, $\overline{AB}$ or $\overline{EF}$?

[b] What is the ratio of $\overline{AB}$ to $\overline{EF}$?

[c] What is the length of $\overline{AB}$ if $\overline{EF}$ is the unit?

Think

A rectangle is 12 in. long and 5 in. wide. Find the dimensions of another rectangle such that the ratio of the perimeter of the second rectangle to the perimeter of the first is 2 to 1 and the ratio of the area of the second to the area of the first is also 2 to 1.

11-3 COMPUTING WITH RATIOS

Many problems that we might ordinarily solve by computing with rational numbers are easier to solve if we think in terms of ratio ideas. The ratio language helps us to visualize sets, and it suggests clever ways of reasoning. For example, suppose we know that a ratio of boys to girls in a group is 2 to 3 and that there are 12 girls. We are to find the number of boys. Using ratio ideas, we can reason:

If there are n boys, the ratio of boys to girls is $n:12$.

$$n:12 = 2:3 \longrightarrow 3 \cdot n = 2 \cdot 12$$
$$3n = 24$$
$$n = 8$$

We can also use fraction notation in solving proportions like the one above:

$$n:12 = 2:3 \longrightarrow \frac{n}{12} = \frac{2}{3} \longrightarrow 3n = 24$$
$$n = 8$$

Exercises

1. Solve the following proportions for n.

[a] $3:2 = 15:n$

[b] $5:3 = 10:n$

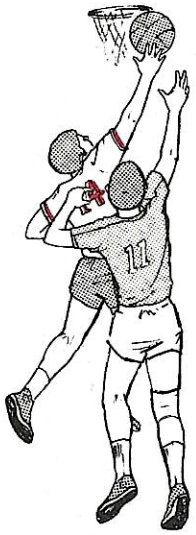
[c] 1 to 5 = n to 100

[d] 3 to n = 75 to 100

[e] $\frac{n}{8} = \frac{12}{16}$

[f] $\frac{25}{n} = \frac{15}{9}$

2. At a picnic, the ratio of adults to children is 3 to 5. If there are 21 adults, how many children are there? Write a proportion and solve.



3. If each 3 boy scouts eat 10 doughnuts, how many doughnuts are needed for 24 boy scouts? Write a proportion and solve.

4. The ratio of field goals scored in a basketball game to field goals attempted is 2:5. Eighty field goals were attempted. How many field goals were scored?

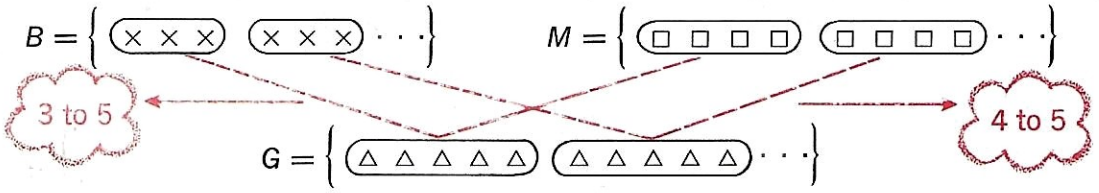
5. The ratio of miles driven to gallons of gas used was 55 to 4. How far was it possible to drive on a 20-gal tank of gas?

6. The ratio of distance in inches shown on a map to actual distance in miles is 1 to 50. If the map distance between two cities is 9 in., how many miles is it from one city to the other?

7. The ratio of apple to peach trees is 5:3. If there are 117 peach trees, how many apple trees are there?

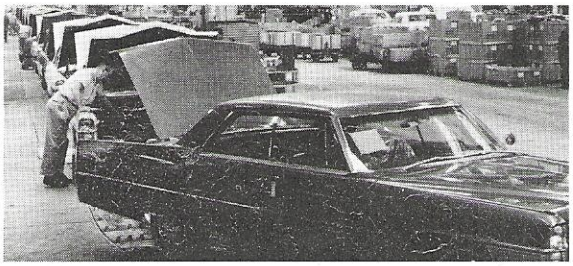
8. The ratio of population of city *A* to city *B* is 12:5. If there are 180,000 people in *A*, how many people are there in *B*?

9. Consider three groups of people, one of boys, one of men, and one of girls.



[a] What is the ratio of boys to girls?
[b] What is the ratio of men to girls?
[c] If the group of boys and men are combined, what is the ratio of this group to the group of girls?
[d] What is the ratio of boys to men?

★ 10. Assembly line *X* produces 5 cars while assembly line *Y* produces 4. Assembly line *Z* produces 3 cars while assembly line *Y* produces 2. Give the ratio that compares the production rate of
[a] line *X* with line *Y*
[b] line *Z* with line *Y*
[c] line *X* with line *Z*

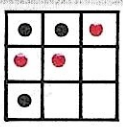


An automobile assembly line

★ 11. In a group of men, women, and children, the ratio of men to children is 1 to 2, and the ratio of women to children is 1 to 3. What ratio compares the total number of men and women with the number of children?

★ 12. The ratio 5:2 compares $\overline{AB}$ to $\overline{XY}$, and 3:2 compares $\overline{CD}$ to $\overline{XY}$.
[a] If $\overline{XY}$ is taken as the unit segment, what is the length of $\overline{AB}$? Of $\overline{CD}$?
[b] If $\overline{CD}$ is chosen as the unit segment, what is the length of $\overline{XY}$? Of $\overline{AB}$?

Think

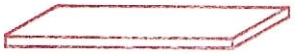


An interesting type of "ticktacktoe" can be played as follows: Each player places his three counters in alternate order until all six counters are down. If no one has won the game at this point, then each player moves a counter (alternating turns) until the game is won. A move consists of sliding one of the counters to an adjacent square.

11-4 RATIO, PROPORTION, AND SCALE DRAWINGS

Many uses of ratio and proportion are related to geometry, such as scale drawings, maps, graphs, and charts. The exercises in this section illustrate some of these applications.

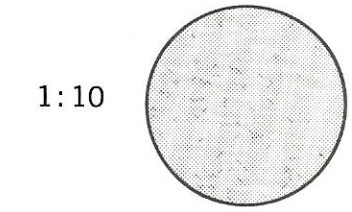
Often, if you want to draw a very large or very small object accurately, you must use a scale. For example, a scale would be necessary to draw a whale or a building accurately, or bacteria, or the small one-celled animal called an **amoeba**. A scale of 1:100 might be used to picture a large object. That is, the dimensions of the picture would be just $\frac{1}{100}$ of the actual dimensions of the object. A scale of 10:1 might be used to picture a small object. The drawing would be 10 times the size of the object. The scale is often shown on the drawing. For example, the drawing represents a board 9 feet long.



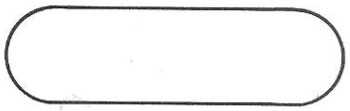
Scale: 1 cm = 3 ft

Exercises

1. The diameter of a softball is about 4 in., and the diameter of a basketball is about 10 in. Does the scale drawing represent a softball or a basketball?



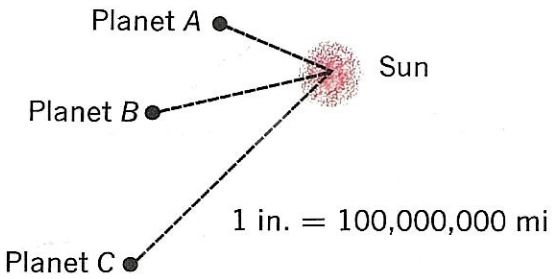
2. The oval figure represents a racetrack. Is the racetrack $\frac{1}{4}$ mi, $\frac{1}{2}$ mi, or 1 mi in circuit?



1:8000

3. The table gives distances from some of the planets to the sun.

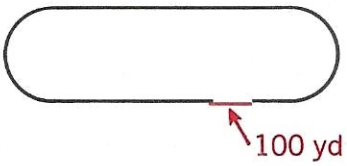
Planet	Distance (miles)
Mercury	36,000,000
Venus	67,000,000
Earth	93,000,000
Mars	142,000,000
Jupiter	483,000,000
Saturn	886,000,000



Tell which planet in the table is represented by

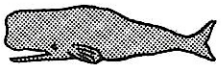
[a] Planet *A* [b] Planet *B* [c] Planet *C*

4. The figure shows a 100-yd stretch measured along a track. Is it a $\frac{1}{2}$ mi, a 1 mi, or a 2-mi track? (There are 1760 yd in 1 mi.)



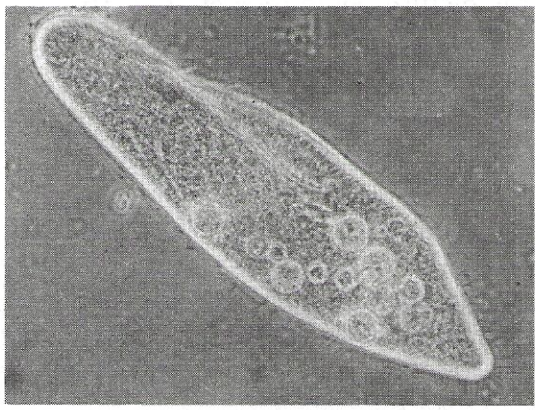
5. In making a scale drawing of a house on a sheet of your notebook paper, would you choose a scale of 1:10, 1:100, or 1:1000?

6. What is the approximate length of the animal shown?



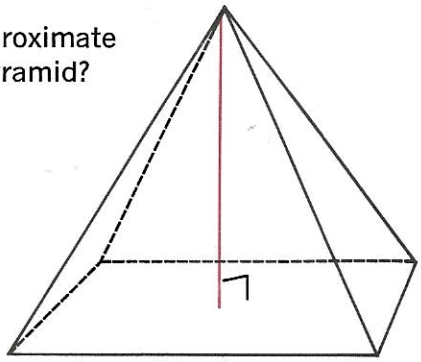
1:1000

7. The creature shown is called a paramecium. What are its approximate dimensions?



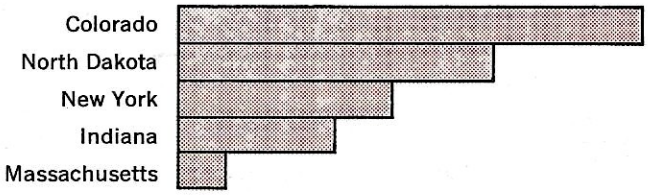
Scale = 250:1

8. What is the approximate height of the pyramid?

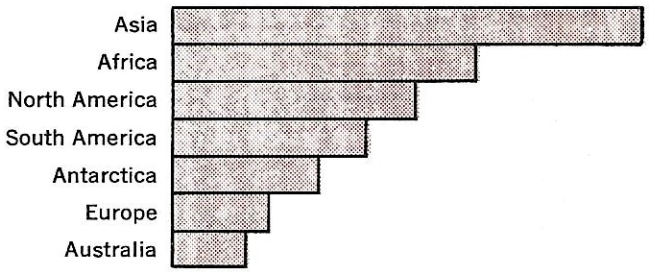


1:5000

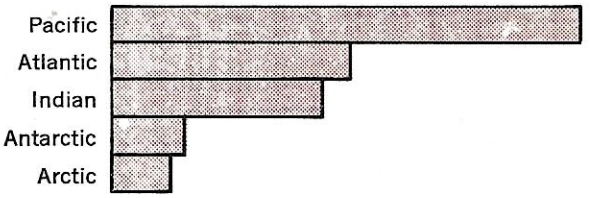
9. The bar graph compares areas of five states. The area of Colorado is about 104,000 sq mi. Estimate the area of each of the other four states.



10. The bar graph compares areas of the seven continents. The area of Asia is about 17,000,000 sq mi. Estimate the area of each of the other continents.



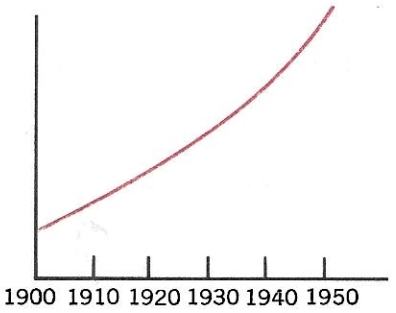
★**11.** The bar graph compares areas of the oceans of the world. The area of the Pacific Ocean is about 64,000,000 sq mi. Estimate the area of each of the other oceans.



★**12.** The chart (line graph) shows the growth in population of the state of Florida from 1900 to 1950.

[a] Estimate the ratios.
1910 population to 1900 population
1920 population to 1900 population
1930 population to 1900 population
1940 population to 1900 population
1950 population to 1900 population

[b] The 1900 population was about 525,000. Estimate the population in 1910, 1920, 1930, 1940, and 1950.



★**13.** Population figures for California at 10-yr intervals are given below.

1900: 1,500,000; 1910: 2,400,000; 1920: 3,400,000
1930: 5,700,000; 1940: 6,900,000; 1950: 10,600,000

To illustrate this information, draw a line graph like the one in Exercise 12.

★**14.** A map of the earth shows the scale 1:500,000,000. Use this figure to calculate the diameter of the earth in miles.

11-5 RATIO AND PROPORTION IN MEASUREMENT

We can use the language of ratio to compare different units of measure.

Linear measures		
1 ft = 12 in.	Ratio of 1 ft to 1 in.: 12 to 1	$\frac{1\text{ ft}}{1\text{ in.}} = \frac{12}{1}$
1 yd = 3 ft	Ratio of 1 yd to 1 ft: 3 to 1	$\frac{1\text{ yd}}{1\text{ ft}} = \frac{3}{1}$
1 mi = 1760 yd	Ratio of 1 mi to 1 yd: 1760 to 1	$\frac{1\text{ mi}}{1\text{ yd}} = \frac{1760}{1}$
Measures of time		
1 min = 60 sec	Ratio of 1 min to 1 sec: 60 to 1	$\frac{1\text{ min}}{1\text{ sec}} = \frac{60}{1}$
1 hr = 60 min	Ratio of 1 hr to 1 min: 60 to 1	$\frac{1\text{ hr}}{1\text{ min}} = \frac{60}{1}$
1 day = 24 hr	Ratio of 1 day to 1 hr: 24 to 1	$\frac{1\text{ day}}{1\text{ hr}} = \frac{24}{1}$
1 wk = 7 days	Ratio of 1 wk to 1 day: 7 to 1	$\frac{1\text{ wk}}{1\text{ day}} = \frac{7}{1}$

Discussion Exercises

1. Explain the steps in each of the following examples.

(A) $\frac{4\text{ in.}}{1\text{ ft}} = \frac{4\text{ in.}}{1 \cdot 12\text{ in.}} = \frac{1}{3}$

(B) $\frac{3\text{ hr}}{40\text{ min}} = \frac{3 \cdot 60\text{ min}}{40\text{ min}} = \frac{9}{2}$

(C) $\frac{7\text{ yd}}{20\text{ in.}} = \frac{7 \cdot 3\text{ ft}}{20\text{ in.}} = \frac{7 \cdot 3 \cdot 12\text{ in.}}{20\text{ in.}} = \frac{63}{5}$

(D) $\frac{3\text{ wk }4\text{ day}}{1\text{ wk }3\text{ day}} = \frac{(3 \cdot 7\text{ day} + 4\text{ day})}{(1 \cdot 7\text{ day} + 3\text{ day})} = \frac{25}{10} = \frac{5}{2}$

2. Give the ratio in simplest terms. Explain your work.

[a] 5 in. to 1 ft
[b] 2 ft to 5 in.
[c] 4 ft to 3 yd
[d] 3 yd to 4 ft
[e] 1 yd to 9 in.
[f] 1 ft 4 in. to 4 in.
[g] 1 yd 1 ft 2 in. to 1 ft
[h] 2 wk 3 day to 1 wk 1 day
[i] 1 min 20 sec to 30 sec
[j] 3 min to 3 sec
[k] 11 min to 11 sec
[l] 43 min to 43 sec

Exercises

1. Copy each exercise, replacing the letters by the correct numbers. You need not reduce ratios to simplest terms.

[a] $\frac{3\text{ ft}}{8\text{ in.}} = \frac{3 \cdot x\text{ in.}}{8\text{ in.}} = \frac{y}{8}$

[b] $\frac{2\text{ yd}}{11\text{ ft}} = \frac{2 \cdot x\text{ ft}}{11\text{ ft}} = \frac{y}{11}$

[c] $\frac{44\text{ ft}}{3\text{ yd}} = \frac{44\text{ ft}}{3 \cdot x\text{ ft}} = \frac{44}{t}$

[d] $\frac{1\text{ mi}}{1\text{ ft}} = \frac{1 \cdot y\text{ ft}}{1\text{ ft}} = \frac{t}{1}$

[e] $\frac{3\text{ min}}{40\text{ sec}} = \frac{3 \cdot x\text{ sec}}{40\text{ sec}} = \frac{y}{40}$

[f] $\frac{3\text{ hr}}{20\text{ sec}} = \frac{3 \cdot x\text{ min}}{20\text{ sec}} = \frac{3 \cdot x \cdot y\text{ sec}}{20\text{ sec}} = \frac{3 \cdot x \cdot y}{20}$

[g] $\frac{3\text{ day}}{75\text{ min}} = \frac{3 \cdot x\text{ hr}}{75\text{ min}} = \frac{3 \cdot x \cdot y\text{ min}}{75\text{ min}} = \frac{3 \cdot x \cdot y}{75}$

[h] $\frac{5\text{ hr}}{4\text{ day}} = \frac{5\text{ hr}}{4 \cdot x\text{ hr}} = \frac{5}{y}$

[i] $\frac{3\text{ ft}}{9\text{ in.}} = \frac{3 \cdot x}{9}$

[j] $\frac{5\text{ hr}}{45\text{ min}} = \frac{5 \cdot x}{45}$

[k] $\frac{4\text{ yd}}{8\text{ in.}} = \frac{4 \cdot x \cdot y}{8}$

[l] $\frac{3\text{ hr}}{90\text{ sec}} = \frac{3 \cdot x\text{ min}}{90\text{ sec}} = \frac{3 \cdot x \cdot y}{90}$

[m] $\frac{2\text{ mi}}{440\text{ ft}} = \frac{2 \cdot x \cdot y}{440}$

[n] $\frac{2\text{ wk}}{35\text{ min}} = \frac{2 \cdot x \cdot y \cdot z}{35}$

2. As we have learned earlier, in metric linear measure, 1 kilometer (km) = 1000 meters (m), 1 m = 100 centimeters (cm), and 1 cm = 10 millimeters (mm). Give the values for x, y, and z.

[a] $\frac{3\text{ m}}{50\text{ cm}} = \frac{3 \cdot x}{50}$

[b] $\frac{3\text{ cm}}{7\text{ mm}} = \frac{3 \cdot x}{7}$

[c] $\frac{7\text{ km}}{850\text{ m}} = \frac{7 \cdot x}{850}$

[d] $\frac{200\text{ m}}{4\text{ km}} = \frac{200}{4 \cdot x}$

[e] $\frac{85\text{ mm}}{15\text{ cm}} = \frac{85}{15 \cdot x}$

[f] $\frac{420\text{ cm}}{8\text{ m}} = \frac{420}{8 \cdot x}$

[g] $\frac{1\text{ m}}{1\text{ mm}} = \frac{x\text{ cm}}{1\text{ mm}} = \frac{x \cdot y}{1}$

[h] $\frac{3\text{ km}}{250\text{ cm}} = \frac{3 \cdot x\text{ m}}{250\text{ cm}} = \frac{3 \cdot x \cdot y}{250}$

[i] $\frac{4\text{ km}}{75\text{ mm}} = \frac{4 \cdot x \cdot y \cdot z}{75}$

In Exercises 3 through 14, give the indicated ratios in simplest terms.

3. 5 cm to 1 m
4. 2 m to 10 cm
5. 1 mm to 1 cm
6. 5 mm to 5 cm
7. 6 cm to 4 m
8. 20 cm to 4 m
9. 2 m to 50 cm
10. 2 m 45 cm to 50 cm
11. 1 km 420 m to 200 m
★**12.** 1 m 23 cm 4 mm to 4 m 32 cm 1 mm
★**13.** 1 km 234 m 56 cm 7 mm to 7 km 654 m 32 cm 1 mm
★**14.** 1 mi 234 yd 56 ft 7 in. to 7 mi 654 yd 32 ft 1 in.

11-6 RATIO AND PERCENT

The word **percent** comes from the Latin phrase **per centum** which means **in the hundred**. When we say, "Six percent of the students are absent," we mean:

Six in each hundred of the students are absent.

Percent language is a special kind of ratio language. The diagram below suggests how the symbol % for percent may have been invented.

$$6:100 \longrightarrow 6/100 \longrightarrow 6/00 \longrightarrow 6\ 0/0 \longrightarrow 6\%$$

In working simple percent problems, it is often helpful to think in terms of ratio ideas.

Discussion Exercises

1. In a group of 500 people, 35 are more than seventy years old.
- [**a**] What is the ratio of the number of people over seventy to the total group?
- [**b**] How many in each 100 are over seventy years old?
- [**c**] What percent of the people are more than seventy years old?

2. On $200 borrowed from a bank for a year, the interest charge was $12.
- [**a**] In simplest terms, what is the ratio of the interest to the amount borrowed?
- [**b**] What was the interest charge per $100?
- [**c**] What percent interest rate was charged?

3. In a city of 1200 inhabitants, 6% of all persons work in a certain factory. How many persons work in the factory?

4. When we say that 7 is 14% of 50, we mean that the ratio of 7 to 50 is 14 to 100. We can find what percent 7 is of 50 by solving the proportion $\frac{7}{50} = \frac{n}{100}$. Solve each proportion and tell what you have found.

[a] $\frac{11}{50} = \frac{n}{100}$ [b] $\frac{3}{50} = \frac{a}{100}$ [c] $\frac{p}{100} = \frac{7}{25}$ [d] $\frac{3}{10} = \frac{x}{100}$ [e] $\frac{9}{20} = \frac{b}{100}$

Exercises

1. Write each statement as a proportion.

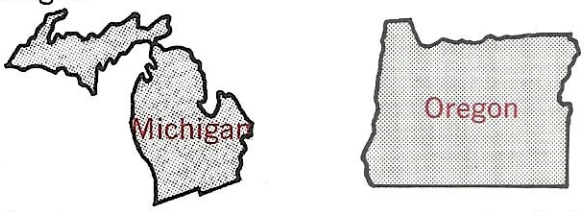
[a] 1 is 50% of 2. (Answer: $\frac{1}{2} = \frac{50}{100}$)
[b] 4 is 25% of 16.
[c] 100 is 10% of 1000.
[d] 5 is 1% of 500.
[e] 25% of 80 is 20.
[f] 5% of 40 is 2.
[g] 15% of 60 is 9.
[h] 250% of 30 is 75.
[i] 80 is 40% of 200.
[j] 30% of 200 is 60.
[k] 33% of 300 is 99.

2. Solve each proportion for n.

[a] $\frac{1}{2} = \frac{n}{100}$ [b] $\frac{4}{10} = \frac{n}{100}$ [c] $\frac{1}{5} = \frac{n}{100}$ [d] $\frac{3}{5} = \frac{n}{100}$ [e] $\frac{1}{4} = \frac{n}{100}$ [f] $\frac{2}{1} = \frac{n}{100}$ [g] $\frac{300}{200} = \frac{n}{100}$ [h] $\frac{28}{400} = \frac{n}{100}$

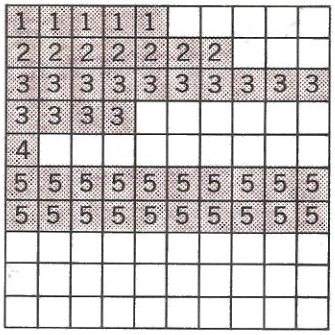
3. In solving the following problems, use your answers for Parts **a** through **d** of Exercise 2.

[a] If 1 of every 2 persons has at least one cold a year, what percent of all people have at least one yearly cold?
[b] If 4 of every 10 trees in an orchard are apple trees, what percent of all the trees are apple trees?
[c] If 1 cup of fruit juice is used in making 5 cups of punch, what percent of the punch is fruit juice?
[d] The areas of Michigan and Oregon are approximately in the ratio 3 to 5. Express the area of Michigan as a percent of the area of Oregon.



4. In solving the following problems, use your answers for Parts **e** through **h** of Exercise 2.

[a] What percent of 4 is 1?
[b] Express 2 as a percent of 1.
[c] What percent is 300 of 200?
[d] 28 is what percent of 400?



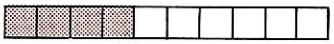
5. Use the diagram to answer the questions.

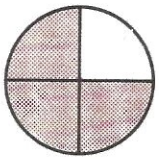
[a] What percent of the squares is marked 1? Marked 2? Marked 3? Marked 4? Marked 5?

[b] What is the ratio of shaded squares to all squares? What percent is shaded?

[c] What is the ratio of unshaded squares to all squares? What percent is unshaded?

6. What percent of the bar is shaded? Unshaded? What is the sum of these percents?





7. What percent of the circle is shaded? Unshaded? What is the sum of these percents?

8. The interest paid on a \$300 loan was \$18. Express the interest as a percent of the loan. (It may help you to write a proportion: $\frac{18}{300} = \frac{n}{100}$.)

9. If 30 tomato plants died out of a total of 600, what percent died?

10. A salesman received a commission of \$40 for making a \$1000 sale. What percent of the sales price did he receive as his commission?

11. A bank charged 6% per year interest on a loan of \$800. At the end of the year, the borrower paid back the loan plus the interest. How much interest did he pay?

12. A man borrowed \$700 for a year. The interest rate was 8% per year. How much interest did he pay on his loan?

13. Find the amount of interest per year.

[a] 200 at 6%	[d] 300 at 12%	[g] 900 at 8%	[j] 4500 at 8%
[b] 700 at 4%	[e] 1000 at 3%	[h] 640 at 5%	[k] 2300 at 1%
[c] 500 at 5%	[f] 250 at 4%	[i] 3000 at 10%	[l] 7500 at 7%

14. A store advertises a sale. All items are to be sold at 30% off the marked prices.

[a] A fur coat is marked \$400. How much less is the sales price?

[b] A suit is marked \$50. For what will it be sold?

★**15.** Find a number n so that

[a] 50% of $n = 40$	[c] 25% of $n = 120$	[e] 1% of $n = 6$
[b] 10% of $n = 15$	[d] 40% of $n = 20$	[f] 300% of $n = 24$

★**16.** After spending 40% of his money at a fair, Jim had \$9.00 left. How much had he originally?

★**17.** Between 1950 and 1960, the population of a city increased by 20%. If the 1960 population was 60,000, what was the 1950 population?

11-7 FROM PERCENTS TO DECIMALS

Ratio language is helpful in finding a percent of a number that is a multiple of 100. To find 7% of \$300, we can think:

3 **hundreds;** 7 **out of each hundred;** 21

However, if we are figuring interest on a sum of money and need to compute 5.25% of \$784.93, ratio ideas are little help. We have used fraction and decimal notation for rational numbers. Percent notation is a third way to represent rational numbers.

We can think of $n\%$ as an abbreviation for the rational number $\frac{n}{100}$ or $n \cdot \frac{1}{100}$.

$$n\% = \frac{n}{100} = n \times \frac{1}{100} = n \times .01$$

$n\%$	$\frac{n}{100}$	$n \times \frac{1}{100}$	$n \times .01$	Decimal
37%	$\frac{37}{100}$	$37 \times \frac{1}{100}$	$37 \times .01$	.37
.3%	$\frac{.3}{100}$	$.3 \times \frac{1}{100}$	$.3 \times .01$	.003
500%	$\frac{500}{100}$	$500 \times \frac{1}{100}$	$500 \times .01$	5
$12\frac{1}{2}\%$	$\frac{12\frac{1}{2}}{100}$	$12\frac{1}{2} \times \frac{1}{100}$	$12\frac{1}{2} \times .01$	.125

Exercises

1. Find a fraction equivalent to each percent, using the fact that $n\% = n \times \frac{1}{100}$.

[a] 7% [c] 240% [e] 400%
[b] $\frac{1}{4}\%$ [d] $2\frac{1}{2}\%$ [f] $12\frac{1}{2}\%$

2. Find a decimal equivalent to each percent, using the fact that $n\% = n \times .01$.

[a] 4% [c] 12.5% [e] .5%
[b] 81% [d] 321% [f] 250%

3. Which is the larger:

[a] $\frac{1}{2}$ or $\frac{1}{2}\%$? [b] .1 or .1%? [c] .2 or 2%? [d] $\frac{1}{10}$ or 1%?

4. Write a lowest-terms fraction for each percent.

[a] 25% [c] 20% [e] 100% [g] $\frac{1}{3}\%$ [i] $12\frac{1}{2}\%$ [k] $33\frac{1}{3}\%$
[b] 75% [d] 50% [f] 2% [h] .3% [j] $6\frac{1}{4}\%$ [l] $66\frac{2}{3}\%$

5. Write each number as a decimal.

[a] 11% [c] 123% [e] 150% [g] .03% [i] 34.2% [k] $87\frac{1}{2}\%$
[b] 24% [d] 1% [f] 42.6% [h] 12.35% [j] 3.42% [l] $8.7\frac{1}{2}\%$

6. Find the products, first replacing each percent by a fraction.

[a] 50% of 84 [c] 10% of 250 [e] 20% of 55 ★[g] $16\frac{2}{3}\%$ of 12
[b] 25% of 48 [d] 15% of 80 [f] 75% of 64 ★[h] $37\frac{1}{2}\%$ of 40

7. Find the products, first replacing each percent by a decimal.

[a] 20% of 340 [c] 5% of 483 [e] 34% of 426 ★[g] $8\frac{1}{2}\%$ of 4
[b] 6% of 45.3 [d] 240% of 6.5 [f] .3% of 54.2 ★[h] 5.4% of 25

★ **8.** Copy and complete the table.

$50\% = \frac{1}{2}$	[b] $3\frac{1}{8}\% =$
$25\% = \frac{1}{4}$	[c] $1\frac{9}{16}\% =$
$12\frac{1}{2}\% = \frac{1}{8}$	[d] $\frac{25}{32}\% =$
[a] $6\frac{1}{4}\% =$	[e] $\frac{25}{64}\% =$

★ **9.** Copy and complete the table.

$\frac{1}{3} = 33\frac{1}{3}\%$	[c] $\frac{1}{48} =$
$\frac{1}{6} = 16\frac{2}{3}\%$	[d] $\frac{1}{96} =$
[a] $\frac{1}{12} =$	[e] $\frac{1}{192} =$
[b] $\frac{1}{24} =$	[f] $\frac{1}{384} =$

★ **10.** Using the table completed for Exercise 8, express each fraction below as a percent.

[a] $\frac{3}{8}$ (**Hint:** $\frac{3}{8} = \frac{1}{4} + \frac{1}{8}$)
[b] $\frac{5}{16}$ (**Hint:** $\frac{5}{16} = \frac{1}{4} + \frac{1}{16}$)
[c] $\frac{5}{8}$ [d] $\frac{7}{32}$ [e] $\frac{9}{16}$ [f] $\frac{11}{32}$
[g] $\frac{11}{16}$ [h] $\frac{9}{64}$ [i] $\frac{7}{64}$ [j] $\frac{17}{128}$
[k] $\frac{15}{128}$ [l] $\frac{13}{16}$ [m] $\frac{17}{32}$ [n] $\frac{41}{64}$

Think

Use six sticks (all the same length) and without breaking any of them, place them in such a way that you obtain exactly four triangles having the same size and shape.

11-8 FRACTION, DECIMAL, AND PERCENT NOTATIONS

The examples show a shortcut for expressing rational numbers as percents. Study these examples and try to discover this shortcut.

A $\frac{3}{4} = 100 \cdot \frac{3}{4} \cdot \frac{1}{100}$

$= 100 \cdot \frac{3}{4}\,\%$

$= 75\,\%$

B $\frac{2}{5} = 100 \cdot \frac{2}{5} \cdot \frac{1}{100}$

$= 100 \cdot \frac{2}{5}\,\%$

$= 40\,\%$

C $\frac{1}{200} = 100 \cdot \frac{1}{200} \cdot \frac{1}{100}$

$= 100 \cdot \frac{1}{200}\,\%$

$= \frac{1}{2}\,\%$

Exercises

1. Copy and complete the table.

Fractions		Decimals (hundredths)	Percents
Lowest terms	Hundredths		
$\frac{1}{2}$	$\frac{50}{100}$	.50	50%
$\frac{1}{4}$		.25	
$\frac{2}{5}$			40%
$\frac{1}{5}$			
			75%
		.60	
	$\frac{30}{100}$		

2. Find the percent for each number.

[a] $\frac{3}{4}$	[e] $\frac{3}{5}$	[i] 1	[m] .35	[q] .06	[u] .120
[b] $\frac{4}{5}$	[f] $\frac{7}{10}$	[j] $2\frac{1}{2}$	[n] .4	[r] .425	[v] .025
[c] $\frac{1}{10}$	[g] $\frac{8}{50}$	[k] 25	[o] 2.15	[s] .775	[w] 3.25
[d] $\frac{3}{10}$	[h] $\frac{11}{25}$	[l] $\frac{31}{100}$	[p] .43	[t] .006	[x] .325

3. To change a decimal to a percent, we multiply by 100. Do you know a shortcut rule for multiplying by 100? You just relocate the decimal point. Use this rule and convert all decimals to percents. Do your work mentally.

[a] .321	[c] .56	[e] 2.425	[g] 2.00	[i] .006	[k] 40
[b] .413	[d] 1.34	[f] 13.16	[h] 4.1	[j] 12	[l] .00025

4. Use the rule of Exercise 3 "in reverse" to change the percents to decimals.

[a] 41.2%	[c] 1.05%	[e] 7%	[g] .0025%	[i] .31%	[k] 35%
[b] 24.3%	[d] 316%	[f] .5%	[h] 5000%	[j] 350%	[l] 3.5%

5. It is sometimes easier to change fractions to percents by first replacing the fractions by decimals.

Example: $\frac{5}{8} \rightarrow \begin{array}{r} .625 \\ 8\overline{)5.000} \end{array}$ $\frac{5}{8} = 62.5\%$

Find the decimal for each fraction and then convert to percent. Round off to the nearest tenth of a percent.

[a] $\frac{3}{8}$ [b] $\frac{7}{8}$ [c] $\frac{5}{16}$ [d] $\frac{5}{12}$ [e] $\frac{1}{6}$ [f] $\frac{5}{6}$ [g] $\frac{1}{9}$ ★[h] $\frac{43}{520}$ ★[i] $\frac{218}{30}$

★**6.** Here is an interesting way to change fractions to percents. First multiply numerator and denominator by 100 and then divide both by the proper number to obtain hundredths.

$$\frac{1}{3} \rightarrow \frac{100}{300} \rightarrow \frac{33\frac{1}{3}}{100} \rightarrow 33\frac{1}{3}\%$$

$$\frac{2}{7} \rightarrow \frac{200}{700} \rightarrow \frac{28\frac{4}{7}}{100} \rightarrow 28\frac{4}{7}\%$$

Use this method to change each fraction to a percent.

[a] $\frac{1}{2}$	[f] $\frac{8}{7}$	[k] $\frac{12}{8}$
[b] $\frac{1}{4}$	[g] $\frac{4}{9}$	[l] $\frac{5}{6}$
[c] $\frac{1}{8}$	[h] $\frac{5}{11}$	[m] $\frac{4}{5}$
[d] $\frac{2}{3}$	[i] $\frac{17}{13}$	[n] $\frac{7}{12}$
[e] $\frac{1}{6}$	[j] $\frac{5}{8}$	[o] $\frac{23}{41}$

Think

There are 12 dentists in a town. In one class of 26 students, each student goes to one of these dentists.

1. Is it possible that some dentist has none of these students as a patient?
2. Is it possible that each student goes to a different dentist?
3. Show that at least three students go to the same dentist.

11-9 USING PERCENT—INTEREST

The table illustrates how percent is used in borrowing (and lending) money.

	Principal	Rate	Interest
	Amount borrowed ($)	Yearly interest rate (%)	Yearly cost of loan ($)
A	300	6	18.00
B	5000	4	200.00
C	600	$3\frac{1}{2}$	21.00
D	100	$12\frac{1}{2}$	12.50
E	1250	5	62.50

Discussion Exercises

1. Explain the following diagram.

Interest = Principal · Rate $\longrightarrow$ $I = P \cdot R$

2. [a] Explain how you would find the total amount to repay at the end of 1 yr if you borrowed money as discussed in the table.

[b] Find these total amounts for each row in the table above.

3. What would the loan in Row A of the table cost for

[a] 2 yr [b] 3 yr [c] $\frac{1}{2}$ yr [d] 5 yr

4. Explain how to find how much you would have to repay in 6 months (mo) for the loan in Row C.

5. How much would you have to repay for the loan in Row B at the end of 3 yr? Explain how you got your answer.

6. Suppose T stands for the number of years. Explain the following formula.

$$I = P \cdot R \cdot T$$

7. Explain how you would find the interest on a loan of $300 at a rate of 2% per month for 5 mo.

It is helpful to think of an interest charge as so many dollars per \$100 borrowed. For example, if you borrow \$800 for 1 yr at a 5% yearly rate, the interest charge is \$5.00 per \$100

8 hundreds are borrowed.
The interest for 1 yr is $8 \cdot \$5.00$ or \$40.

Exercises

1. An interest rate of $3\frac{1}{2}\%$ per yr is a charge of \$3.50 per \$100. Express each interest rate as cost per \$100 borrowed per year.

[a] 7% per yr
[b] 3% per mo
[c] 6% per yr
[d] 5% per mo
[e] 10% per yr
[f] 8% per mo
[g] $5\frac{1}{2}\%$ per yr
[h] $1\frac{1}{4}\%$ per mo
[i] $8\frac{3}{4}\%$ per yr
[j] $\frac{3}{4}\%$ per mo
[k] $1\frac{3}{4}\%$ per mo
[l] 4.2% per mo

2. Find the interest for 1 yr on a loan of

[a] \$200 at 6% per yr
[b] \$600 at 7% per yr
[c] \$500 at 5% per yr
[d] \$800 at 4% per yr
[e] \$300 at 4% per yr
[f] \$200 at 8% per yr
[g] \$700 at 2% per yr
[h] \$600 at 9% per yr
[i] \$400 at $6\frac{1}{4}\%$ per yr
[j] \$300 at $5\frac{1}{2}\%$ per yr
[k] \$1000 at $8\frac{3}{4}\%$ per yr
[l] \$1200 at $\frac{3}{4}\%$ per yr

3. For each part of Exercise 2, find the interest if the money is borrowed for 2 yr.

4. Find the interest on each loan.

[a] \$200 borrowed for 3 mo at 2% per mo
[b] \$400 borrowed for 5 mo at 1% per mo
[c] \$600 borrowed for 8 mo at $1\frac{1}{2}\%$ per mo
[d] \$800 borrowed for 4 mo at $\frac{3}{4}\%$ per mo

5. Give the yearly rates equivalent to the following monthly rates.

[a] 2% [b] $1\frac{1}{2}\%$ [c] $2\frac{1}{4}\%$ [d] 1% [e] $\frac{3}{4}\%$ [f] $1\frac{3}{4}\%$

6. Give the monthly rates equivalent to the following yearly rates.

[a] 12% [b] 18% [c] 6% [d] 8% [e] 9% [f] 30%

7. Find the interest, using the formula $I = P \cdot R$.

[a] \$456 borrowed 1 yr at 4% per yr
[b] \$2146 borrowed 1 yr at 5% per yr
[c] \$755 borrowed 1 mo at 2% per mo
[d] \$542 borrowed 1 mo at 1% per mo

8. Find the interest, using the formula $I = P \cdot R \cdot T$.

[a] \$355 for 2 yr at 5% per yr
[b] \$456 for $\frac{1}{2}$ yr at 6% per yr
[c] \$644 for $1\frac{1}{2}$ yr at 8% per yr
[d] \$155 for 3 mo at 2% per mo
[e] \$468 for 5 mo at $1\frac{1}{2}\%$ per mo
[f] \$540 for $3\frac{1}{2}$ mo at 1% per mo

9. Find the interest rate if the interest on

[a] \$200 borrowed for 1 yr is \$10
[b] \$400 borrowed for 1 yr is \$28
[c] \$700 borrowed for 1 yr is \$14
[d] \$600 borrowed for 1 yr is \$36
[e] \$300 borrowed for 1 yr is \$30
[f] \$400 borrowed for 2 yr is \$24
[g] \$200 borrowed for 3 yr is \$36
[h] \$500 borrowed for 5 yr is \$100

★ **10.** Find the interest rate.

[a] Interest for 1 yr, \$12; principal, \$240
[b] Interest for 1 yr, \$21; principal, \$350
[c] Interest for 1 yr, \$22.50; principal, \$450
[d] Interest for 1 yr, \$14.40; principal, \$320
[e] Interest for 1 yr, \$40; principal, \$640

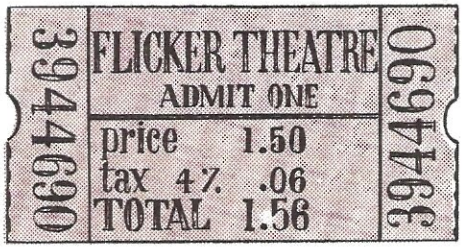
★ **11.** Find the principal.

[a] Interest for 1 mo, \$12; rate, 2% per mo
[b] Interest for 1 mo, \$4.50; rate, 1% per mo
[c] Interest for 1 mo, \$5.40; rate, $1\frac{1}{2}$% per mo
[d] Interest for 1 mo, \$5.50; rate, 2% per mo

★ **12.** A man borrowed \$500 for six mo. At the end of that time, he repaid principal and interest. If he paid a total of \$515, what interest rate was he paying?

11-10 OTHER APPLICATIONS OF PERCENT

The figures illustrate some of the many uses of percent.

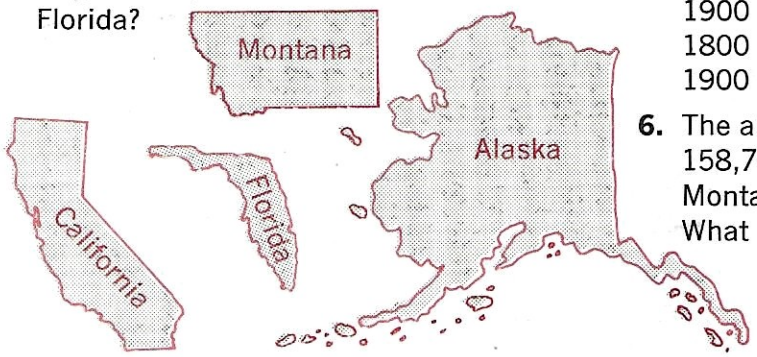


Exercises, Short Stories

1. Sales $3200. Commission 2% of sales. How much commission earned?
2. Price of car $2200. Sales tax 3%. How much tax?
3. Population 6000 in 1950. Increase in next 10 years 17%. What is 1960 population?
4. Income tax 20% on first $5000 of income, 22% on next $4000. Income $9000. How much tax?
5. Price $180. 2% discount for cash. How much off for cash?
6. Marked price $80. Sale price 40% off. What is sale price?
7. Marked price $60. Sale price $48.
   [a] Saved how much due to the sale?
   [b] What percent of the marked price is this saving?
8. Population 8 million in 1950. Population 10 million in 1960.
   [a] How many more people in 1960?
   [b] What was the percent increase?
9. Sales $10,000,000 in 1950. Sales $8,000,000 in 1960. What percent decrease?
10. Suit of clothes $80. Sales tax $2.40. What percent sales tax?

PROBLEMS—PERCENT

1. Water expands about 10% when it freezes. If 800 cu in. of water freeze, how much ice is formed?
2. Ice contracts about 9% when it melts. If 880 cu in. of ice melt, how much water is formed? (Compare your result with Exercise 1.)
3. The area of Alaska is about 586,000 sq mi. If the area of Florida is about 10% of this, what is the area of Florida?



4. Florida is the largest state east of the Mississippi. The smallest state is Rhode Island. If the area of Rhode Island is about 2.07% of the area of Florida, what is Rhode Island's area? (Use Exercise 3.)

5. The United States population in 1800 was about 5.3 million. If the 1900 population was 1430% of the 1800 population, what was the 1900 population?
6. The area of California is about 158,700 sq mi. The area of Montana is about 7.28% less. What is the area of Montana?

7. The largest fresh water lake in the world is Lake Superior whose area is 31,800 sq mi. The second largest is Lake Victoria, the source of the Nile river. The area of Lake Victoria is 8.44% less than the area of Lake Superior. What is the area of Lake Victoria?

8. The largest continent, Asia, has an area of about 17 million sq mi. The area of the smallest continent, Australia, is about 3 million. Describe Australia's area as a percent of Asia's.

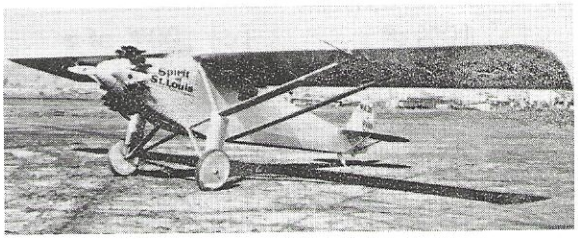
★ **9.** The 1960 population of Asia was about 1870 million. The 1960 population of Australia was about 11 million. The population of Asia is what percent of the population of Australia?

★ **10.** Lindbergh's flight from New York to Paris in 1927 took about 33 hr 30 min. In 1961, Major W. R. Payne in a Hustler jet bomber flew from New York to Paris in about 9.9% of this time. What was Major Payne's time?

★ **11.** The population of the world in 8000 B.C. is estimated to have been about 5 million. The predicted population for 2000 A.D. is 6 billion. What percent is the 8000 B.C. population of the predicted 2000 A.D. population?

★ **12.** On July 7, 1965, Allen (Philadelphia) had 100 hits in 296 times at bat; Mays (San Francisco) had 93 hits in 277 times at bat. Which one led the National League in batting? Find each player's percent of hits.

★ **13.** On July 7, 1965, Davalillo (Cleveland) had 98 hits in 296 times at bat. Yastrzemski (Boston) had 69 hits in 203 times at bat. Which one led the American League in batting?



Lindbergh's plane, the *Spirit of St Louis*

Think

A used-car dealer bought a car from a wholesaler and marked the price up 50% from what he paid. When the car didn't sell in two months, he had a 40%-off sale.

1. Did he make or lose money?
2. How much did he make or lose if he paid $600 for the car originally?

11-11 ESTIMATION

The art of estimating accurately is one of the most useful skills in mathematics, and estimation involving percent is one of the most common types of problems. As before, estimation with percents involves selecting approximations for the numbers before doing the mental computation.

Exercises

Exercises 1 through 5 should help you with the estimation exercises that follow them.

1. If $x = 56$, find each amount.

[a] 10% of x [b] 1% of x [c] 100% of x [d] .1% of x [e] .01% of x [f] 1000% of x [g] 10,000% of x [h] .001% of x

2. If $y = 28.3$, find each amount.

[a] 10% of y [b] 1% of y [c] 100% of y [d] .1% of y [e] .01% of y [f] 1000% of y [g] 10,000% of y [h] .001% of y

3. If $z = 354.2$, find each amount.

[a] 10% of z [b] 1% of z [c] 100% of z [d] .1% of z [e] .01% of z [f] 1000% of z [g] 10,000% of z [h] .001% of z

4. If $w = 268.45$, find each amount.

[a] 10% of w [b] 1% of w [c] 100% of w [d] .1% of w [e] .01% of w [f] 1000% of w [g] 10,000% of w [h] .001% of w

5. If $p = 21.3$, find each amount.

[a] 1% of p [b] 2% of p [c] 3% of p [d] 10% of p [e] 20% of p [f] 30% of p [g] .1% of p [h] .2% of p [i] .4% of p [j] 100% of p [k] 200% of p [l] 500% of p

6. Find the best estimate for each exercise.

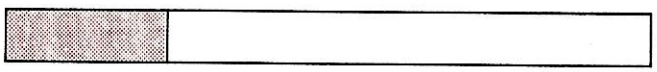
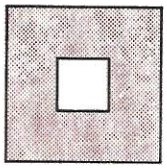
[a] 3% of 369 (100, 10, 1000, .1, .01)
[b] 27% of 46 (12, 1.2, 120, .12, .012)
[c] 98% of 3.7 (.37, .037, 37, 3.7, 370)
[d] 16% of 284 (.45, .045, 450, 4.5, 45)
[e] 49% of 68.4 (34, 3.4, .34, 340, .034)
[f] 33% of 200 (660, 6.6, 66, .66, .066)
[g] .9% of 167 (167, 1.67, 16.7, .167, 1670)

7. Give the number for ***a***. Then give the number for ***b***.
[a] Since 50 is ***a***% of 100, 49 is about ***b***% of 97.
[b] Since 25 is ***a***% of 100, 27 is about ***b***% of 103.
[c] Since 37 is ***a***% of 100, 36.8 is about ***b***% of 97.
[d] Since 25 is ***a***% of 50, 26.4 is about ***b***% of 51.3.
[e] Since 6 is ***a***% of 10, 5.8 is about ***b***% of 9.4.
[f] Since 30 is ***a***% of 50, 29 is about ***b***% of 51.
[g] Since 10 is ***a***% of 40, .11 is about ***b***% of .39.
[h] Since 16 is ***a***% of 32, 1.6 is about ***b***% of 31.7.

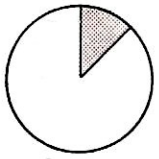
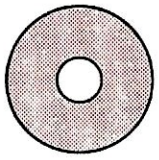
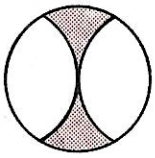
8. Jim was asked to estimate what percent 18.5 is of 18,543. He estimated that it was about 10%. This is a very poor estimate. Which of the estimates below is better?
[a] 1% [b] .1% [c] .01%

9. Joe estimated that .1% of 4021 is about 400. Joe got a zero for this estimate. Which of the following estimates do you consider best?
[a] 4000 [b] 40 [c] 4 [d] .4

10. Estimate what percent of each rectangular region is shaded.

[a]  [b] 

11. Estimate what percent of each circular region is shaded.

[a]  [b]  [c] 

12. George estimated that 24.2 is about 200% of 1214. Now George's teacher keeps him after school to work on arithmetic. What is your estimate?
[a] 20% [b] 2% [c] .2% [d] .02%

13. ***n*** is between 510 and 520. Give estimates for each amount.
[a] 200% of ***n*** [b] 20% of ***n*** [c] 2% of ***n*** [d] .2% of ***n***

14. What number most nearly tells what percent 435 is of 1413?
[a] 300% [b] 30% [c] 3% [d] .3% [e] .03%

15. Pick the number closest to 2% of 79,534.12.
[a] 160,000 [b] 16,000 [c] 1600 [d] 160 [e] 16

★**16.** In 1924, a United States plane circled the earth in 363.1 hr (flying time). An astronaut circles the earth in about 1.4 hr. Estimate the percent that the astronaut's time is of the 1924 plane's time.

★**17.** The area of Alaska is about 586,400 sq mi. The area of Rhode Island is about 1214 sq mi. Estimate the area of Alaska as a percent of the area of Rhode Island.

★**18.** The distance from the earth to the sun is about 93,000,000 mi. The distance from the earth to the moon is about 240,000 mi. Estimate the moon's distance as a percent of the sun's distance.

★**19.** World population is about 3,500,000,000. Australia's population is about 11,000,000. Estimate what percent the world population is of Australia's.

★**20.** The velocity of light is about 669,600,000 mph. The velocity of a satellite is about 18,000 mph. Estimate the satellite's velocity as a percent of the velocity of light.

CHAPTER REVIEW

Key Concepts

1. When we use a fraction, we are dealing with ideas of **ratio.** When we write a fraction like $\frac{3}{5}$, we are comparing two sets. For every 3 things in one set, there are 5 in the other.
2. The language of ratio helps us to visualize clearly the relationships between sets described by rational numbers. It suggests ways of thinking that are helpful in solving problems.
3. We can compare ratios as we compare rational numbers. The ratios 2 to 3 and 4 to 6 are equal because $\frac{2}{3} = \frac{4}{6}$.
4. A statement that two ratios are equal is a **proportion.** In the proportion $2:3 = 4:6$, the numbers 2 to 6 are called the **extremes** and the numbers 3 and 4 are called the **means.**
5. Most problem solving with ratios involves setting up proportions and solving for a missing number.

$$\frac{n}{12} = \frac{2}{3} \longrightarrow 3n = 24$$

In solving proportions, we use the fact that the product of the means is equal to the product of the extremes.

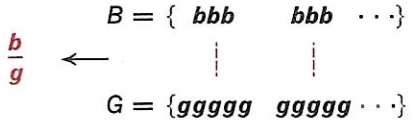
6. Many problems that we might ordinarily work by adding, multiplying, or dividing rational numbers are easier to solve if we use the language of ratio and visualize the sets that are involved.

Vocabulary List

extremes	interest rate	principal	scale
graph	means	proportion	simplest terms
interest	percent	ratio	

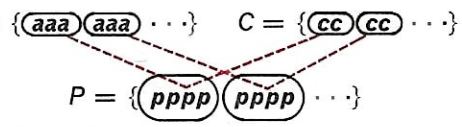
Review Exercises

1.



[a] What fraction does the diagram suggest?
[b] Give four different symbols for the ratio of boys (B) to girls (G).
[c] If there are 30 girls, how many boys are there?
[d] What is the ratio of the number of girls to total number of boys and girls?
[e] What is the ratio of number of girls to number of boys?
[f] If there are 40 more girls than boys, how many girls are there?

2.



Give the ratio.
[a] Apple (A) to peach trees (P)
[b] Peach to cherry trees (C)
[c] Apple to cherry trees

3. One baseball team has won 12 games and lost 9. Another team has won 15 and lost 12. Which team has the better record?

4. Express each ratio in simplest terms.
[a] 12:8 [b] 20:50 [c] 35:25 [d] 16:4

5. For the proportion, $3:8 = 15:40$, tell
[a] which numbers are the means
[b] which are the extremes

6. Solve the proportions.

[a] $\frac{2}{5} = \frac{n}{20}$ [c] $n:6 = 20:8$

[b] $\frac{3}{n} = \frac{4}{12}$ [d] $4:15 = 12:n$

7. From the six ratios below, three proportions can be formed. Write these proportions.

[a] 1:2 [c] 25:15 [e] 40:80
[b] 15:9 [d] 30:24 [f] 45:36

Check in each case that the product of the means is equal to the product of the extremes.

8. The scale for a drawing of a house is 1 to 80. The living room is pictured on the drawing as a rectangle 2 by 3 in. Give the actual length and width of the living room.

9. If a television set priced at \$300 is reduced in price by \$24, what percent is it reduced?

10. Solve each proportion and express the given fraction as a percent.

[a] $\frac{1}{10} = \frac{n}{100}$ [d] $\frac{17}{50} = \frac{n}{100}$
[b] $\frac{5}{4} = \frac{n}{100}$ [e] $\frac{4}{3} = \frac{n}{100}$
[c] $\frac{3}{8} = \frac{n}{100}$ [f] $\frac{5}{7} = \frac{n}{100}$

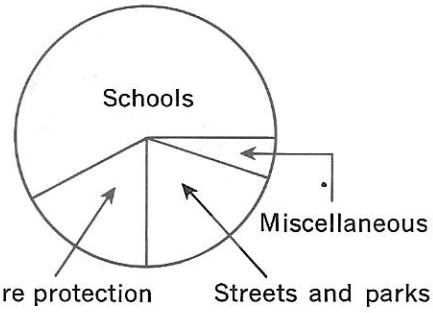
11. Find the interest on each loan for 1 yr.
[a] \$200 at 7% per yr
[b] \$450 at 6% per yr
[c] \$225 at 8% per yr
[d] \$1200 at 5% per yr
[e] \$600 at $4\frac{1}{2}$% per yr

12. A suit of clothes marked \$60 is sold for \$45. By what percent was the price reduced?

13. Mr. Smith's taxes in 1964 were \$800. His 1965 taxes were \$960. What was the percent of increase in his taxes?

14. A furniture salesman receives (in addition to his salary of \$100 per week) a 1% commission on all sales. If his sales for a week were \$1800, what was his income for that week?

15. The circle graph pictures how a certain city spent its taxes. Estimate the percent of all tax money spent for each category.



16. Express each number as a fraction in lowest terms.

[a] 50% [c] 75% [e] 2%
[b] 30% [d] 80% [f] $\frac{1}{2}$%

17. Express each number as a decimal.

[a] 23% [c] 250% [e] .01%
[b] 1.5% [d] .1% [f] .003%

18. Express each number as a percent.

[a] .35 [c] 1.25 [e] .007
[b] .04 [d] .7 [f] 5

19. Express each ratio in simplest terms.

[a] 220 yd to 1 mi [c] 880 yd to 1 mi
[b] 440 yd to 1 mi [d] 165 ft to 220 yd

20. Express each ratio in simplest terms.

[a] 300 cm to 5 m [c] 240 mm to 12 cm
[b] 20 dm to 3 m [d] 4000 m to 1 mm

21. The diameter of the earth is about 8000 mi. The table below gives the approximate ratios of the diameter of other planets to the diameter of earth. Find the approximate diameter of each planet.

Planet	Ratio
Mercury	19:50
Mars	1:2
Neptune	17:5
Saturn	9:1
Pluto	9:20
Venus	9:10
Uranus	56:15
Jupiter	11:1

★**22.** The French franc is worth 20% as much as the American dollar.

[a] What is the value in dollars of 500 French francs?
[b] How many francs could you get in exchange for $50?

★**23.** A Swiss franc is worth about 23% as much as a dollar.

[a] What is the value in dollars of 500 Swiss francs?
[b] About how many francs could you get in exchange for $50?

CHAPTER SUPPLEMENT

Reading List

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Activities

A. Go to a store that is having a sale. Choose three items that are on sale and from the marked price and the rate of discount determine the sale price of each article.

B. Determine the height of your school flagpole by setting up a proportion between your height (measured) and your shadow (measured) and the height (unknown) and shadow (measured) of the flagpole.

C. Study how a dictionary uses scale drawings and report your findings to the class.

D. Interview an architect or engineer in your neighborhood to find out how he constructs and uses blueprints.

E. Using the scale 1 in. = 5 ft, make a plan of your house by measuring the actual dimensions of each room and converting them to the scale size.

F. Keep a record of all your grades in math class over a two-week period. Then determine your average in the form of a percent.

G. How many students in your math class have first names that begin with vowels? What percent of the class is this?

. . . In a minute or two the Caterpillar took the hookah out of its mouth, and yawned once or twice, and shook itself. Then it got down off the mushroom, and crawled away into the grass, merely remarking, as it went, "One side will make you grow taller, and the other side will make you grow shorter."

"One side of what? *The other side of* what?" *thought Alice to herself.*

"Of the mushroom," said the Caterpillar, just as if she had asked it aloud; and in another moment it was out of sight.

12

Geometry Unit IV—Circles

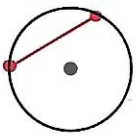
A. NONMETRIC GEOMETRY

12-1 ANGLES AND ARCS IN CIRCLES

A **circle** is a set of points all of which are equally far from a given point called the **center.**

The examples illustrate some important geometric figures related to a circle.

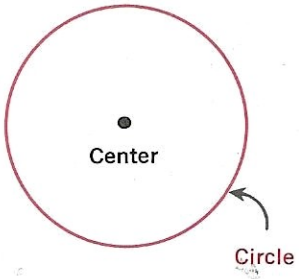
Chord



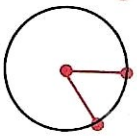
A line segment with endpoints on the circle

Arc

A "part" of the circle

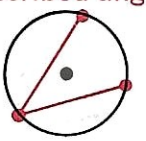


Central angle

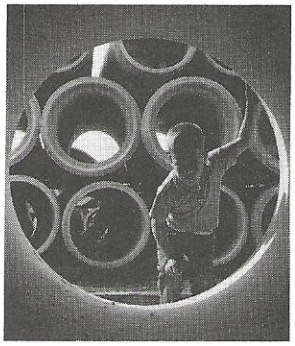


An angle with its vertex at the center of the circle

Inscribed angle



An angle that contains three points of the circle and has its vertex on the circle



12

Geometry Unit IV—Circles

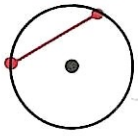
A. NONMETRIC GEOMETRY

12-1 ANGLES AND ARCS IN CIRCLES

A **circle** is a set of points all of which are equally far from a given point called the **center**.

The examples illustrate some important geometric figures related to a circle.

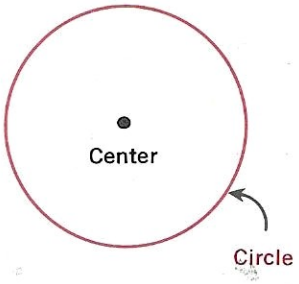
Chord



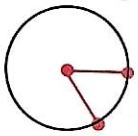
A line segment with endpoints on the circle

Arc

A "part" of the circle

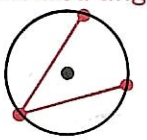


Central angle

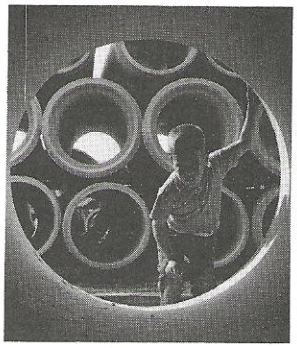


An angle with its vertex at the center of the circle

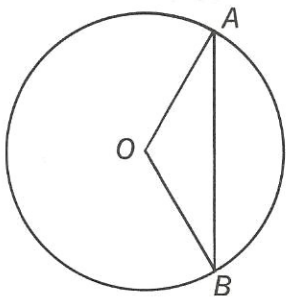
Inscribed angle



An angle that contains three points of the circle and has its vertex on the circle



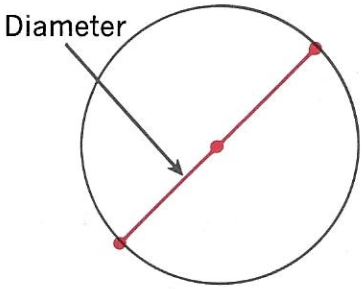
An $\angle AOB$, a chord $\overline{AB}$, and an arc $\overset{\frown}{AB}$ are shown in the figure to the right. Each of these figures is said to **subtend** the other. For example:



(A) $\overset{\frown}{AB}$ subtends $\angle AOB$, or $\angle AOB$ is subtended by $\overset{\frown}{AB}$.

(B) $\angle AOB$ subtends $\overset{\frown}{AB}$, or $\overset{\frown}{AB}$ is subtended by $\angle AOB$.

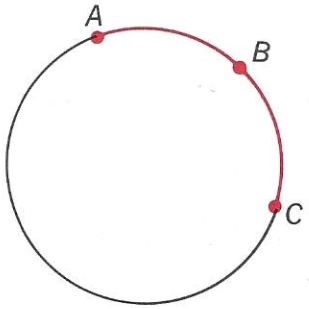
Discussion Exercises



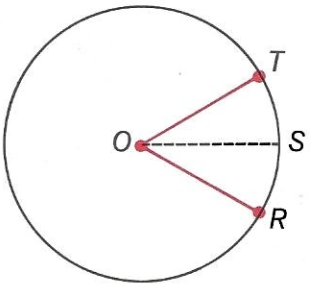
1. A chord that passes through the center of a circle is called a **diameter**. Is there more than one such chord?

2. If $\overline{AB}$ and $\overline{CD}$ are both diameters of the same circle, where do they intersect?

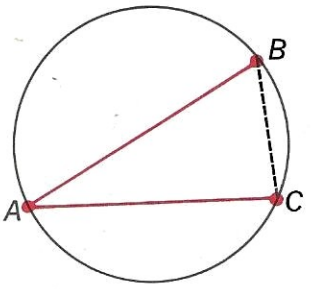
3. The arc illustrated in the figure is arc ABC, written $\overset{\frown}{ABC}$. An arc that is half a circle is called a **semicircle**. What can you say about points A and C if $\overset{\frown}{ABC}$ is a semicircle?



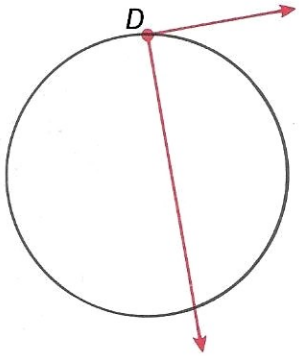
4. [a] Angle TOR is what kind of angle with respect to the circle?
[b] Name three different arcs shown in the figure.
[c] What angle subtends $\overset{\frown}{SR}$?



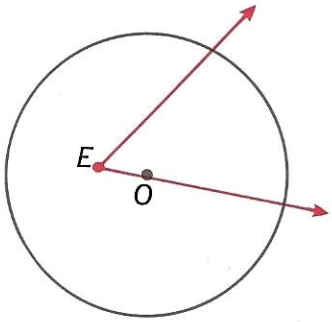
5. [a] Angle BAC is what kind of angle with respect to the circle?
[b] What arc subtends $\angle BAC$?
[c] What chord subtends $\angle BAC$?



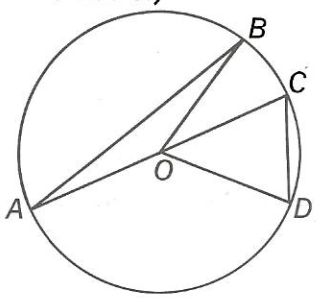
6. Why is $\angle D$ not an inscribed angle?



7. Why is $\angle E$ not a central angle?



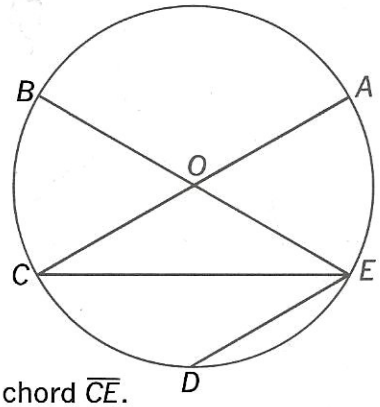
8. [a] Name four central angles. (Point O is the center of the circle.)
[b] Name two inscribed angles.
[c] Name two angles that subtend $\overset{\frown}{BC}$.
[d] What angle subtends $\overline{CD}$?
[e] What kind of triangle is $\triangle DOC$?
[f] Name three chords.
[g] Which chord is a diameter?



Exercises

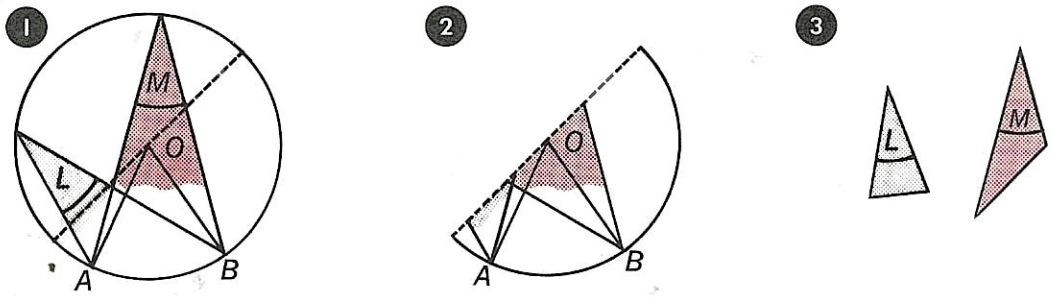
1. Draw a circle and then a chord which is not a diameter. Draw the central angle that subtends the chord. Draw an inscribed angle that subtends the chord.

2. [a] Name the central angle subtended by $\overset{\frown}{AE}$.
[b] Name the central angle subtended by $\overset{\frown}{BC}$.
[c] Name the inscribed angle subtended by $\overset{\frown}{BC}$.
[d] Name the central angle subtended by $\overset{\frown}{CDE}$.
[e] Name two chords that are not diameters.
[f] Name a chord that is a diameter.
[g] Name an inscribed angle that does not have one side a diameter.



[h] Name the central angle that is subtended by the chord $\overline{CE}$.
[i] Name the arc subtended by $\angle BOA$.
[j] Name the arc subtended by $\angle DEB$.

3. Draw a figure with two inscribed angles and one central angle all subtending the same arc. Then cut your paper along the dotted line.
Cut out $\angle L$ and $\angle M$. How do they compare?
Fit $\angle L$ and $\angle M$ into $\angle AOB$. How do they fit?

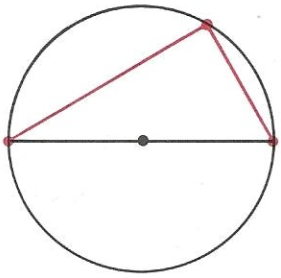


4. Draw a circle and a diameter.

[a] Inscribe an angle that subtends the diameter. What kind of angle is it?

[b] Inscribe another angle subtending the same diameter. What kind of angle do you have this time?

[c] If an inscribed angle subtends a diameter, what kind of angle is it?



5. Draw a circle and label the center O. Draw two diameters and label their endpoints A, B, C, and D. Draw the chords that connect the endpoints of the diameters.

[a] Find four pairs of congruent triangles.

[b] Name a pair of parallel chords.

[c] Find a pair of congruent inscribed angles that subtend the same arc.

[d] Name the rectangle.

[e] Name an arc that is a semicircle and the right angle that subtends it.

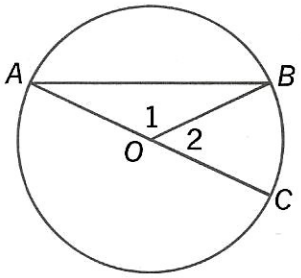
★**6.** [a] What kind of triangle is $\triangle AOB$?

[b] What can you say about $\angle A$ and $\angle B$?

[c] Angles 1 and 2 together make up what kind of angle?

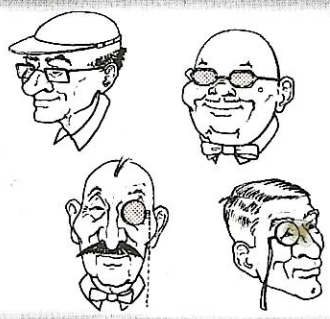
[d] Angles A, B, and 1 together make up what kind of angle?

[e] What does the answer to Part **d** tell you about the relation between $\angle 2$ and the sum of $\angle A$ and $\angle B$?



Think

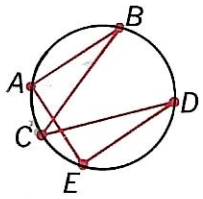
In a certain village, everyone wears corrective lenses (some clear, some tinted). Half of the inhabitants wear monocles, and half of the remaining people do not wear tinted lenses. Also, half of the monocle wearers do not wear tinted lenses. If 18 tinted lenses are enough for all the people who wear them, what is the population of the village?



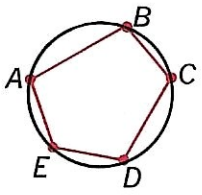
12-2 POLYGONS AND CIRCLES

A polygon is **inscribed** in a circle if each of its vertices is on the circle.

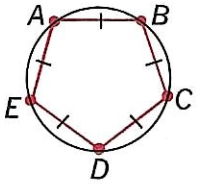
Inscribed pentagons *ABCDE*



Not simple: it crosses itself.



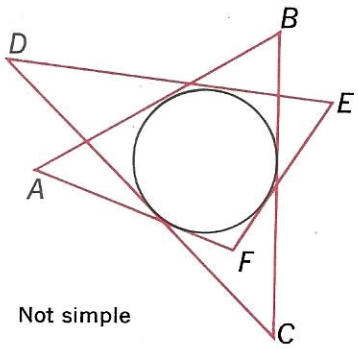
Simple, but not regular



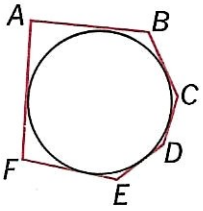
Regular

A polygon is **circumscribed** about a circle if each of its sides touches the circle in just one point.

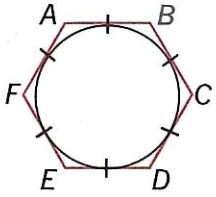
Circumscribed hexagons *ABCDEF*



Not simple



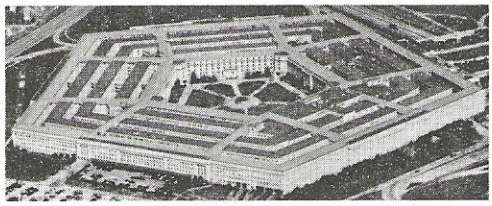
Simple, but not regular



Regular

While most of your work in geometry will deal only with simple polygons, it is important to note that there are other types.

Exercises



Aerial view of the Pentagon Building, Arlington, Virginia

1. Draw four circles. Draw the following simple inscribed polygons, one in each circle: a triangle, a quadrilateral, a pentagon, and a hexagon.

2. Draw a circle and mark eight points on it. Use these points to draw an inscribed octagon that is not simple.

3. Draw a circle and then a simple circumscribed quadrilateral.

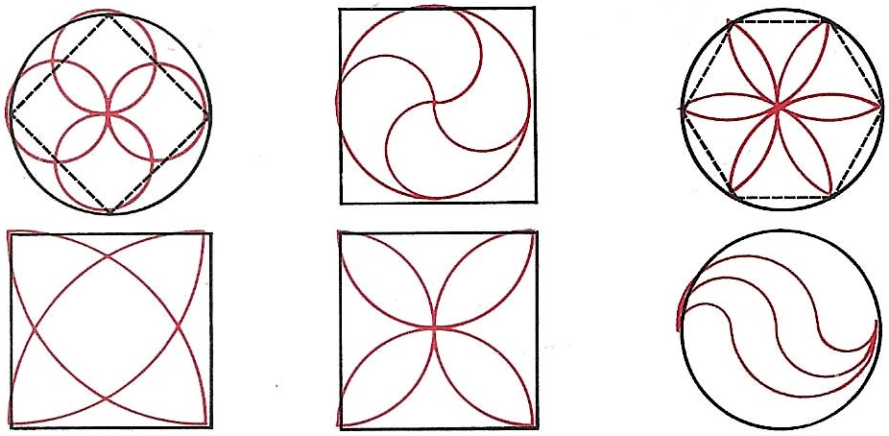
4. Draw a circle and then a circumscribed pentagon that is not simple.

5. Draw a circle and two diameters that are not perpendicular. Label their endpoints E, F, G, and H. Connect the endpoints. What kind of figure is $EFGH$?

6. What kind of figure would you have had in Exercise 5 if the diameters had been perpendicular? Is it a regular polygon?

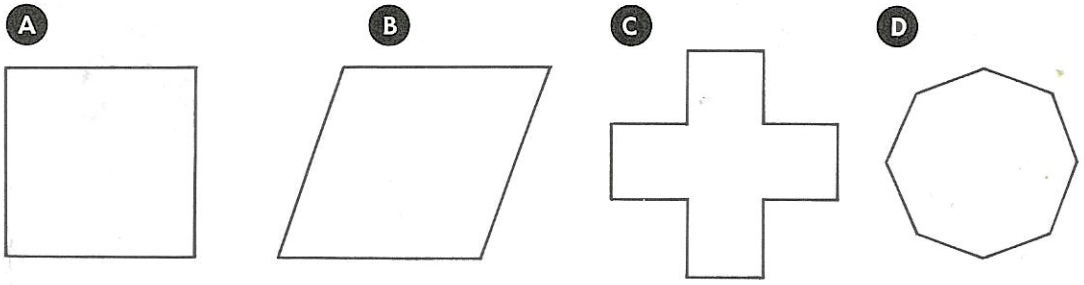
7. Experiment with your ruler and compass to see if you can discover a way to construct a regular inscribed hexagon. (**Hint**: You need six points equally spaced about the circle.)

8. By experimenting with your ruler and compass, you can invent many interesting designs. The figures illustrate a few. Try to copy one of these. Invent some of your own.



9. [a] Which of these simple polygons, with all sides congruent, can be inscribed in a circle?

[b] Which ones are regular?



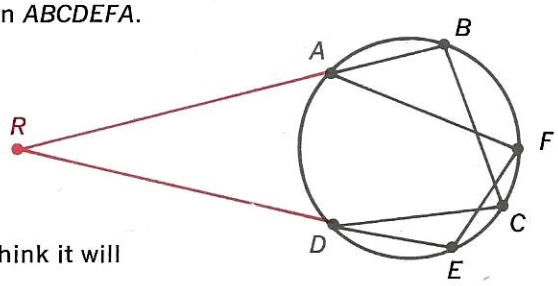
10. It is easy to start with a circle and then draw an inscribed triangle. It is more difficult to start with a triangle and then draw the circle so that the triangle is inscribed. Draw a triangle. Construct the perpendicular bisector of each of two sides. The intersection of these two lines is the center of the circle in which the triangle is inscribed. Draw this circle.

★**11.** Mark seven points around a circle and label them in order *A*, *B*, *C*, *D*, *E*, *F*, and *G*. Draw the polygon *ACEGBDFA*. What figure do you get? Would the same way of skipping vertices work for an octagon?

Think

This construction, called the **magic hexagon**, was discovered by the French mathematician and philosopher Blaise Pascal (1623–1662). Choose any six points on a circle. Name these points, **in any order**, *A*, *B*, *C*, *D*, *E*, and *F*. Draw the hexagon *ABCDEFA*.

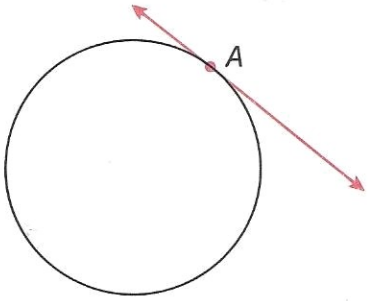
If $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ meet at *R*, $\overleftrightarrow{BC}$ and $\overleftrightarrow{EF}$ meet at *S*, and $\overleftrightarrow{CD}$ and $\overleftrightarrow{FA}$ meet at *T*, can you guess how *R*, *S*, and *T* will be situated?



Draw the picture and see. Do you think it will always work this way? Try it again.

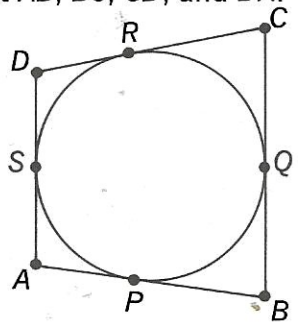
12-3 TANGENTS TO CIRCLES

A line is **tangent** to a circle if it intersects the circle in just one point. In the figure, the line is tangent to the circle at point A. Point A is called the **point of tangency**.



Discussion Exercises

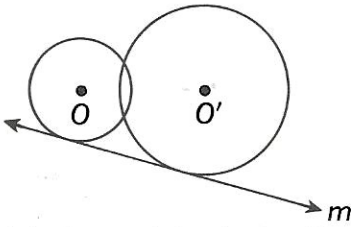
1. Quadrilateral $ABCD$ is circumscribed about the circle. What can you say about $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{DA}$?



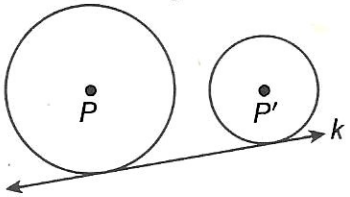
2. Line m is tangent to circles O and O'.

[a] Describe the location of a second line n tangent to both circles.

[b] What can you say about m and n if O and O' are the same size?



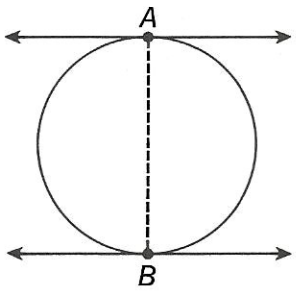
3. Line k is tangent to circles P and P'. Describe the location of three more lines that are tangent to both circles.



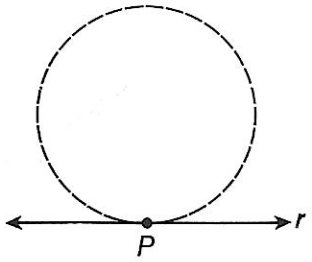
4. [a] How many lines are tangent to a circle at a point on the circle?

[b] How many circles are tangent to a line at a point on the line?

5. Suppose a circle is tangent to each of two parallel lines at points A and B. What do you think is true of $\overline{AB}$?



6. How would you construct a circle tangent to a given line r at a given point P on the line?

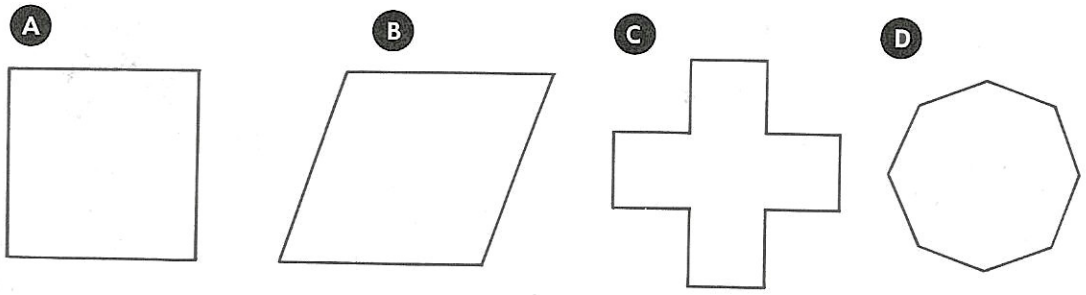


7. [a] What is meant by a line tangent to a sphere?

[b] What is meant by a plane tangent to a sphere?

9. [a] Which of these simple polygons, with all sides congruent, can be inscribed in a circle?

[b] Which ones are regular?



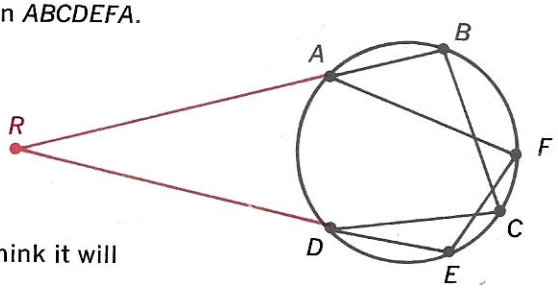
10. It is easy to start with a circle and then draw an inscribed triangle. It is more difficult to start with a triangle and then draw the circle so that the triangle is inscribed. Draw a triangle. Construct the perpendicular bisector of each of two sides. The intersection of these two lines is the center of the circle in which the triangle is inscribed. Draw this circle.

★11. Mark seven points around a circle and label them in order *A*, *B*, *C*, *D*, *E*, *F*, and *G*. Draw the polygon *ACEGBDFA*. What figure do you get? Would the same way of skipping vertices work for an octagon?

Think

This construction, called the **magic hexagon**, was discovered by the French mathematician and philosopher Blaise Pascal (1623–1662). Choose any six points on a circle. Name these points, **in any order**, *A*, *B*, *C*, *D*, *E*, and *F*. Draw the hexagon *ABCDEFA*.

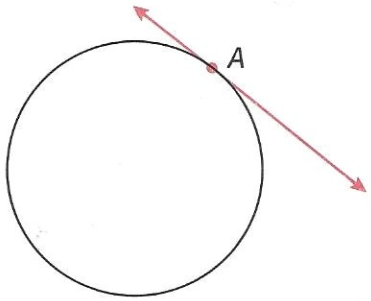
If $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ meet at *R*, $\overleftrightarrow{BC}$ and $\overleftrightarrow{EF}$ meet at *S*, and $\overleftrightarrow{CD}$ and $\overleftrightarrow{FA}$ meet at *T*, can you guess how *R*, *S*, and *T* will be situated?



Draw the picture and see. Do you think it will always work this way? Try it again.

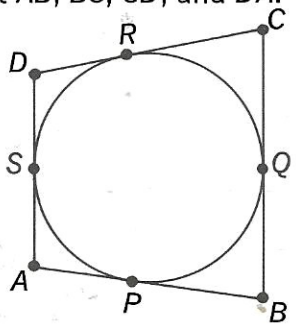
12-3 TANGENTS TO CIRCLES

A line is **tangent** to a circle if it intersects the circle in just one point. In the figure, the line is tangent to the circle at point A. Point A is called the **point of tangency**.



Discussion Exercises

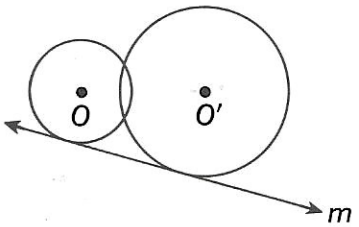
1. Quadrilateral $ABCD$ is circumscribed about the circle. What can you say about $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{DA}$?



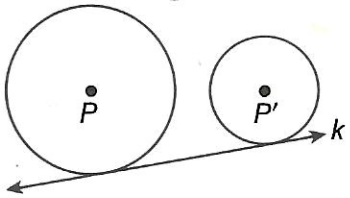
2. Line m is tangent to circles O and O'.

[a] Describe the location of a second line n tangent to both circles.

[b] What can you say about m and n if O and O' are the same size?



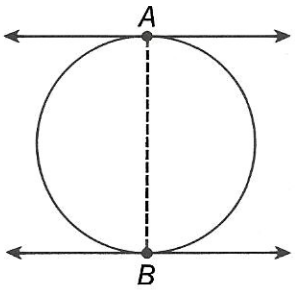
3. Line k is tangent to circles P and P'. Describe the location of three more lines that are tangent to both circles.



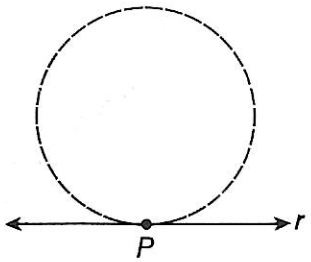
4. [a] How many lines are tangent to a circle at a point on the circle?

[b] How many circles are tangent to a line at a point on the line?

5. Suppose a circle is tangent to each of two parallel lines at points A and B. What do you think is true of $\overline{AB}$?



6. How would you construct a circle tangent to a given line r at a given point P on the line?

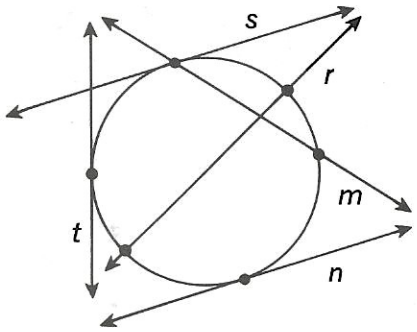


7. [a] What is meant by a line tangent to a sphere?

[b] What is meant by a plane tangent to a sphere?

Exercises

1. Which of the lines are tangent to the circle?

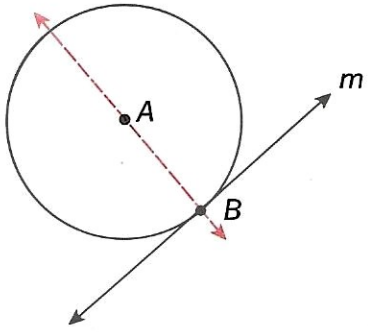


2. Draw a figure showing two different circles, both tangent to a line at the same point.

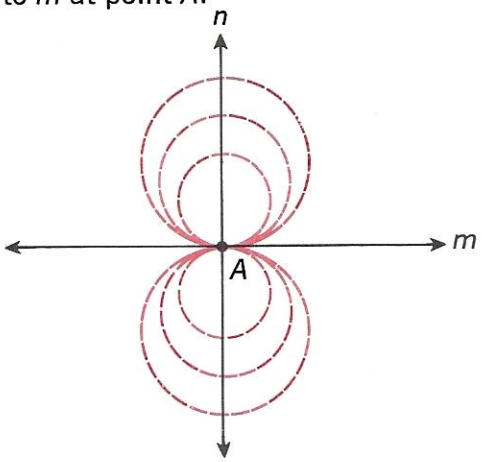
3. Line m is tangent to the circle with center A.

[a] How are the two lines m and $\overleftrightarrow{AB}$ related?

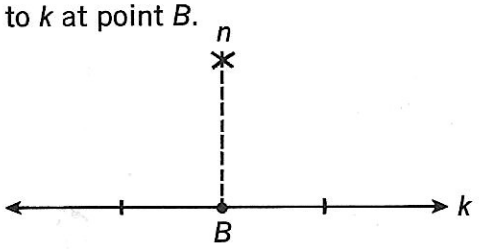
[b] A line passing through the point of tangency and the center of the circle is _?_ to the tangent line.



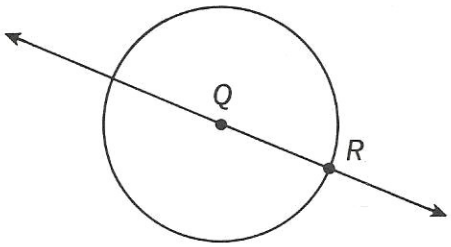
4. Lines m and n are perpendicular to each other. Where are all the centers of the circles that are tangent to m at point A?



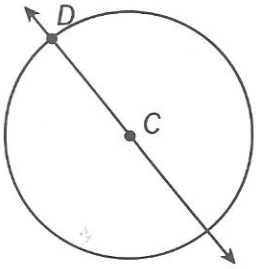
5. Draw a line k. Mark a point B on k. Construct the perpendicular to k at B. Now draw three circles tangent to k at point B.



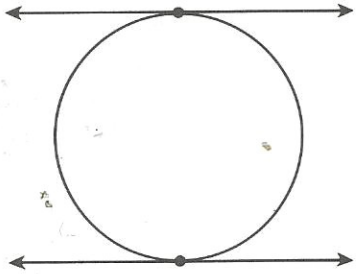
6. Point R is on a circle with center Q. Think about $\overleftrightarrow{QR}$ and the line tangent to the circle at R. How are these lines related?



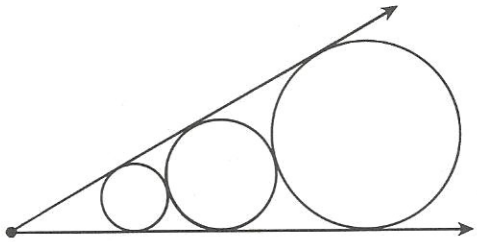
7. Draw a circle and mark a point D on the circle. Label the center C. Draw a line through C and D. Construct the perpendicular to $\overleftrightarrow{CD}$ at D. How is this line related to the circle?



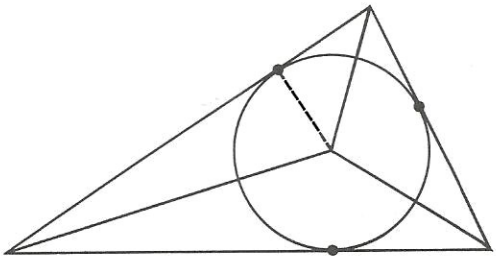
★**8.** Draw two parallel lines. Construct a circle that is tangent to both lines.



★**9.** Draw an acute angle. Construct three different circles that are tangent to the sides of the angle.

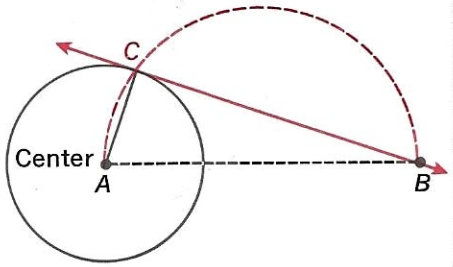


★**10.** Draw a triangle. Bisect each angle. Then complete the construction for the circle inscribed in this triangle.



Think

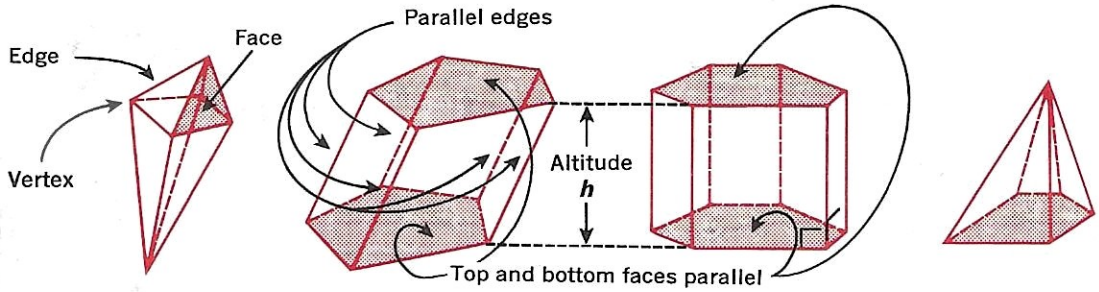
Here is a circle with a center at A and a point B outside it. The dashed circle has diameter $\overline{AB}$.



1. What kind of a figure is $\overset{\frown}{ACB}$?
2. Angle ACB is an inscribed angle of the dashed circle. What is its measure?
3. Is $\overleftrightarrow{CB}$ perpendicular to $\overline{AC}$? Why?
4. What can you say about $\overleftrightarrow{CB}$ with respect to the circle A?

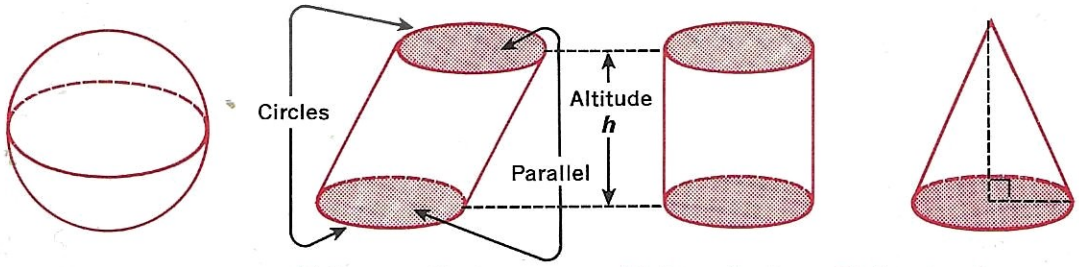
12-4 SPACE FIGURES

A **polyhedron** is the surface of a solid whose faces are polygonal regions.



Octahedron Oblique prism Right prism Pyramid

Some other space figures are rounded. These have circles in them.



Sphere Oblique cylinder Right cylinder Right circular cone

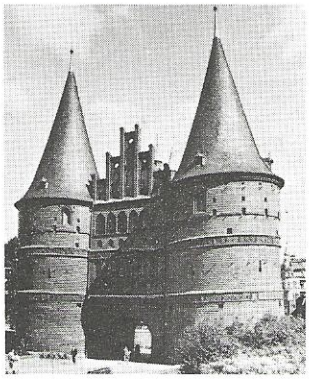
Discussion Exercises

1. What are some of the ways a prism differs from an octahedron?

2. Explain why the oblique prism above is specifically called a **pentagonal prism**.

3. In what way does an oblique prism differ from a right prism?

4. The pyramid pictured above is a pentagonal pyramid because the base is a pentagon. Describe
   [a] a square pyramid [c] a hexagonal pyramid
   [b] a triangular pyramid [d] an octagonal pyramid



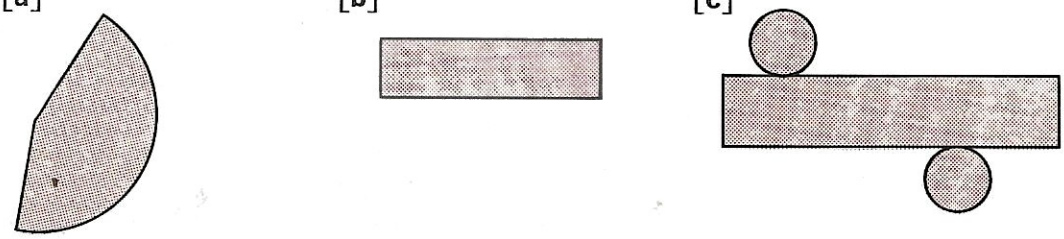
This monument, built in 1477 in Lübeck, Germany, shows the use of the cylinder and the cone in architecture.

5. Each face of a polyhedron is what kind of a geometric figure?

6. What does it mean for two faces of a polyhedron to be parallel?

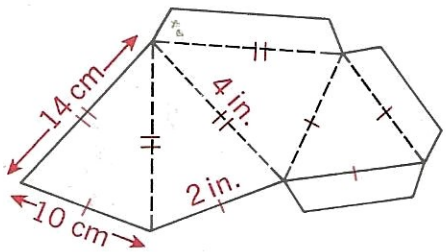
7. Describe the space figures you will get by joining some of the edges for each figure.

[a] [b] [c]

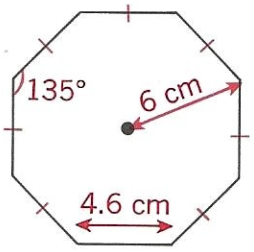


Exercises

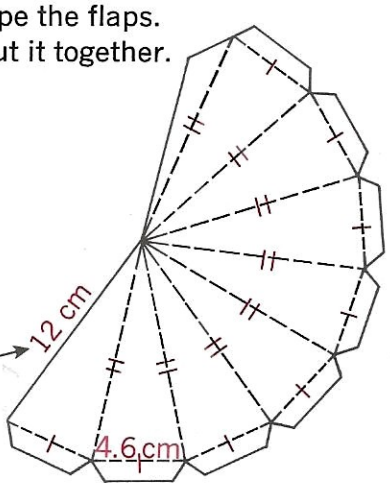
1. Form the indicated space figures by drawing patterns with the dimensions given. Fold on the dotted lines and glue or tape the flaps. Try to visualize what the figure will look like before you put it together.

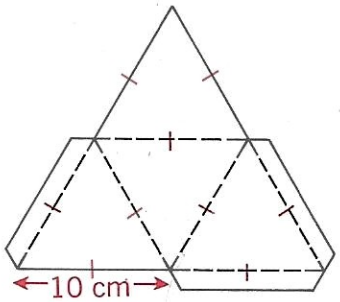


Triangular pyramid

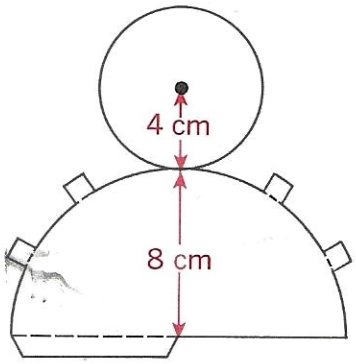


Octagonal pyramid

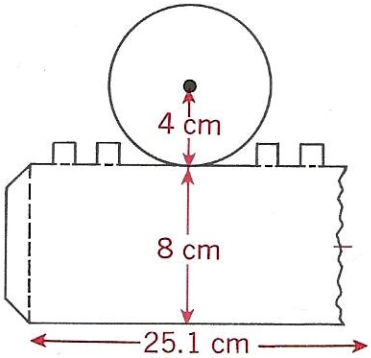




Tetrahedron



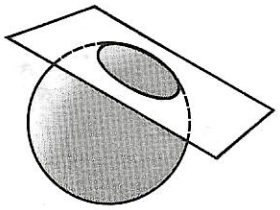
Cone



Cylinder

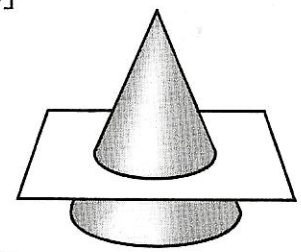
2. Sketch each cross section formed by the intersection of the plane and the space figure. The answer to Part a is given.

[a]

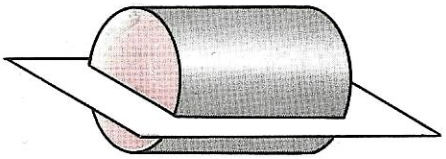


Answer:

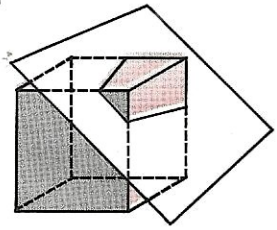
[b]



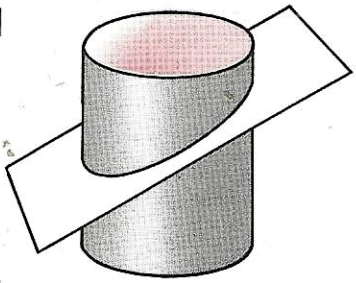
[c]



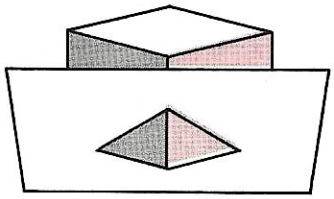
[d]



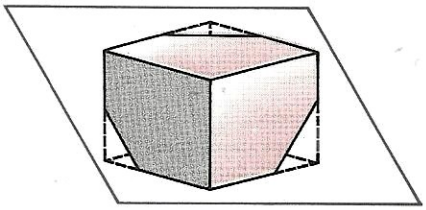
[e]



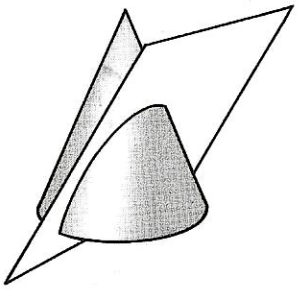
[f]



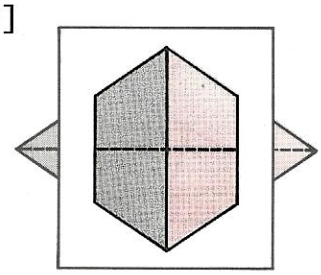
★[g]



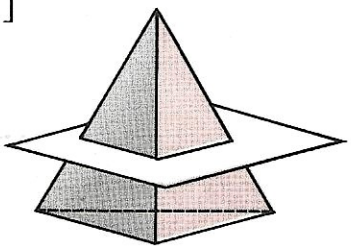
[h]



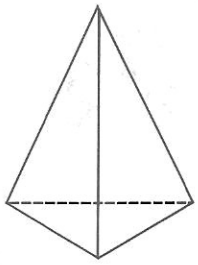
[i]

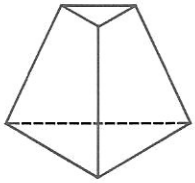


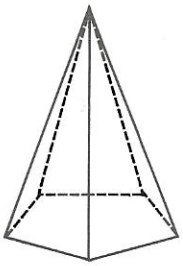
[j]

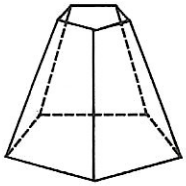


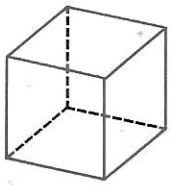
3. Copy and complete the table for Polyhedrons A through H. The letter V stands for number of vertices, F for number of faces, and E for number of edges.

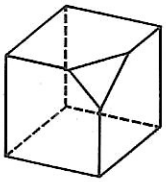
A 

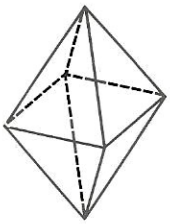
B 

C 

D 

E 

F 

G 

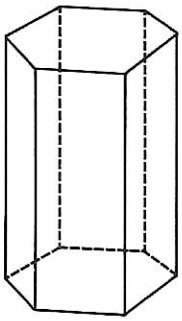
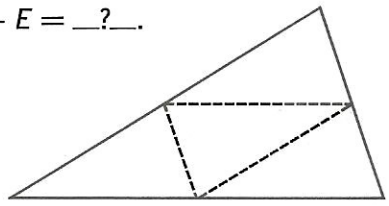
H 

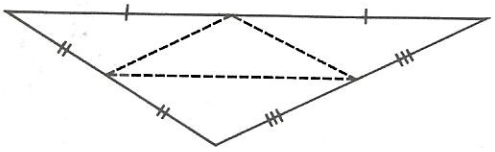
Figure	V	F	E	$V+F$	$(V+F)-E$
A	4	4	6	8	2
B	6				
C					
D					
E					
F					
G					
H					

In general, for polyhedrons such as these, $(V + F) - E =$ _?_.



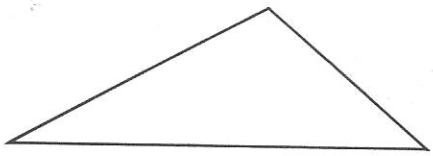
4. Draw an acute triangle. Find the midpoints of the sides and fold along the dotted lines. This should give you a tetrahedron.

★**5.** If you were to copy this figure, could you fold along the dotted lines to make a tetrahedron? Explain your answer.



Think

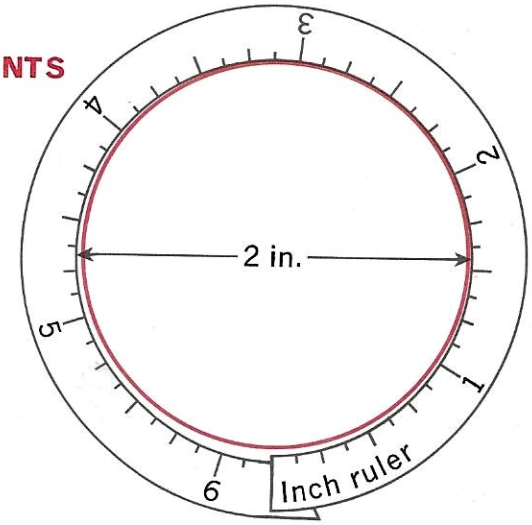
Draw an obtuse triangle. Show how to fold it to make a tetrahedron.



B. METRIC GEOMETRY

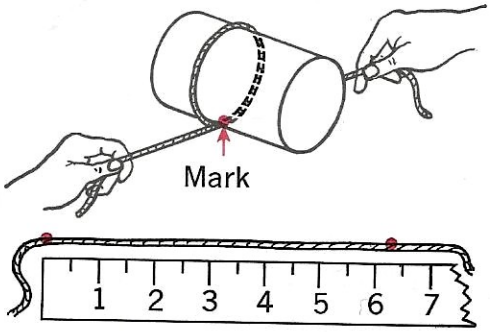
12-5 CIRCUMFERENCE–EXPERIMENTS

If you had a "curved" ruler like the one in the figure, you might be able to measure the circumference of some circles. But this would not be practical or easy. In this lesson and the next, you will learn how to find the distance around a circle using your "straight" ruler.



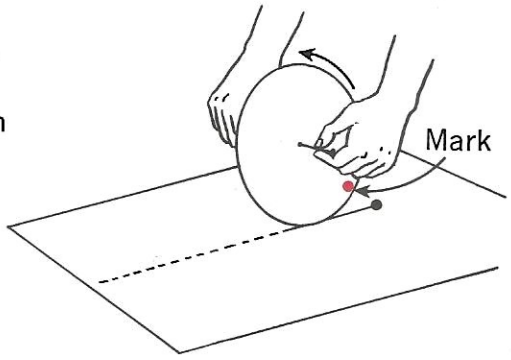
Discussion Exercises

1. Use a piece of string to measure the circumferences of several tin cans, bottles, pipes, or wheels. Also measure their diameters. Then record your findings in a table like the one completed for the picture.



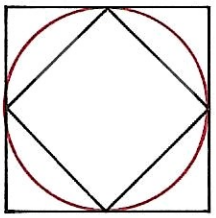
C = circumference	d = diameter	$C \div d$
6.3	2	3.15

2. On stiff cardboard, draw a circle with a 10-cm diameter and then cut it out. Mark a point on the circumference and, using a pin through the center, roll the circle along a straight line. Measure the distance along the line for one full turn and record your findings in the table you made for Exercise 1. Repeat this procedure with circles of diameters 6, 12, 16, and 20 cm.

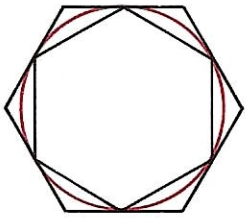


Exercises

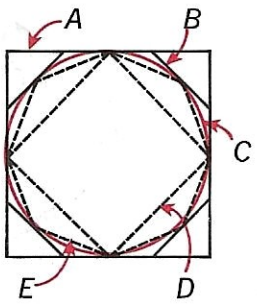
1. [a] Which square has the greater perimeter?
[b] Which is greater, the perimeter of the large square or the circumference of the circle?
[c] Which is greater, the perimeter of the small square or the circumference of the circle?



2. [a] Which hexagon has the greater perimeter?
[b] The circumference of the circle is between the perimeter of the small hexagon and the __?__.

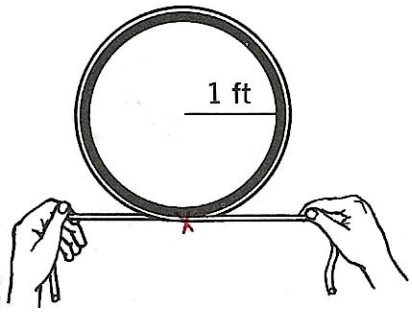


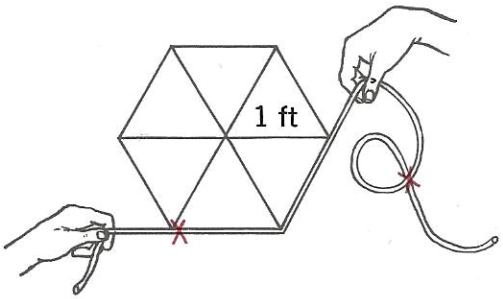
3. Look at the Square A and the Octagon B outside the Circle C. Look also at the Square D (dotted) and the Octagon E (dotted) inside the figure.
[a] Which is greater, the perimeter of A or the perimeter of B?
[b] Which is greater, the perimeter of D or the perimeter of E?
[c] Which is greater, the circumference of C or the perimeter of B?
[d] Which is greater, the circumference of C or the perimeter of E?



4. Use the picture from Exercise 3, thinking of A, B, C, D, and E as distances.
[a] Of the squares in the picture, C is less than A and greater than __?__.
[b] Of the octagons in the picture, C is less than __?__ and greater than __?__.

5. Joe wrapped a cord around the circular pipe shown below and marked the cord. Then he wrapped it around the hexagonal box, also shown. Did he have any cord left?





6. You can estimate the circumference of a circle by finding the __?__ of circumscribed and inscribed polygons.

7. From the discussion exercises, you found that the ratio of the circumference to the diameter ($C \div d$) of any circle is very close to what whole number?

12-6 CIRCUMFERENCE—THE NUMBER π

The table that you made in Section 12–5 suggests that the ratio $C \div d$ is the same number for all circles. This number, the **ratio of circumference to diameter**, is named by the Greek letter π (pronounced "pie").

$$\frac{C}{d} = \pi$$

For thousands of years, mathematicians have been interested in calculating the value of π. In 2000 B.C., the Egyptians used this estimate for π: $\longrightarrow \pi \approx 4 \cdot \left(\frac{8}{9}\right)^2$.

The Bible (I Kings 7:23) implies that π is equal to 3. Archimedes, the Greek mathematician, showed that π is between $3\frac{10}{71}$ and $3\frac{10}{70}$.

Today there are many formulas for computing π. Recently, π was computed on an electronic digital computer to more than 100,000 decimal places. The first fifteen places are $\pi = 3.141592653589792\ldots$.

> And he made a molten sea ten cubits from one brim to the other: it was round all about, and its height was 5 cubits, and a line of 30 cubits did compass it round about.
>
> (I Kings 7:23, King James Version)

More useful approximations for π are

$$\pi \approx 3.1416 \approx 3.14 \approx \frac{22}{7}$$

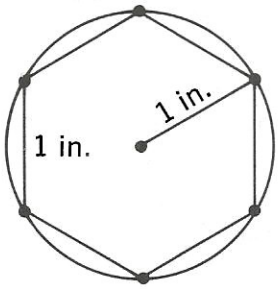
Exercises

1. Use the fact that $\pi \approx \frac{22}{7}$ to compute the approximate circumference of a circle whose diameter is

[a] 14 [b] $\frac{7}{11}$ [c] $\frac{21}{2}$ [d] $\frac{3}{5}$ [e] $2\frac{7}{8}$ [f] $1\frac{3}{11}$ [g] $4\frac{1}{5}$

2. Use the fact that $\pi \approx 3.14$ to find the approximate circumference for each circle in Exercise 1.

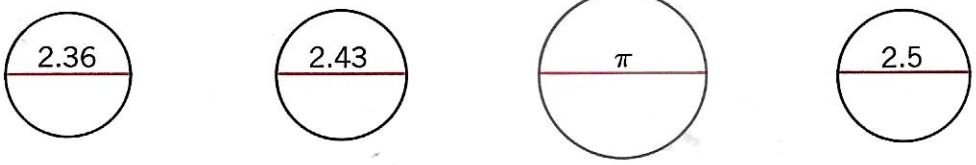
3. The figure shows a circle with a 1-in. radius and an inscribed hexagon with 1-in. sides.



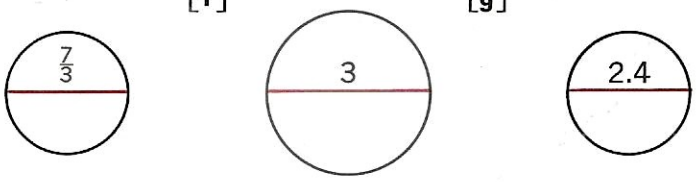
[a] What is the diameter of the circle?
[b] Which is greater, the circumference of the circle or the perimeter of the hexagon?
[c] What is the perimeter of the hexagon?
[d] What is the ratio of the perimeter of the hexagon to the diameter of the circle?
[e] Since the circumference of the circle is "close" to the perimeter of the hexagon, the ratio of the circumference of the circle to its diameter is slightly more than what whole number?

4. List the circles below in order from smallest to largest circumference.

[a] [b] [c] [d]



[e] [f] [g]



5. A wheel has a diameter of about 42 in.

[a] About how many feet will it travel in 1 revolution?
[b] About how many feet will it travel in 10 revolutions?
[c] About how many feet will it travel in 120 revolutions?
★[d] About how many revolutions will it make in traveling 1 mi?

6. [a] Which number, 3.14 or $3\frac{1}{7}$, is greater?
[b] Which number, 3.14 or $3\frac{1}{7}$, is closer to 3.14159?

7. [a] If the diameter of a circle is 1 unit, then its circumference is $\_?\_ \times \pi$.

[b] If the diameter of a circle is 2 units, what is its circumference (exactly)?

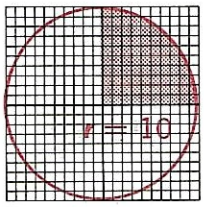
[c] If the diameter of a circle is $\frac{3}{\pi}$ units, what is its circumference (exactly)?

[d] If the diameter of a circle is $\frac{2.54}{2 \times \pi}$ units, what is its circumference (exactly)?

★ **8.** A circular cylinder has a circumference of 13.66 cm. What is its diameter? Use $\pi \approx 3.14$.

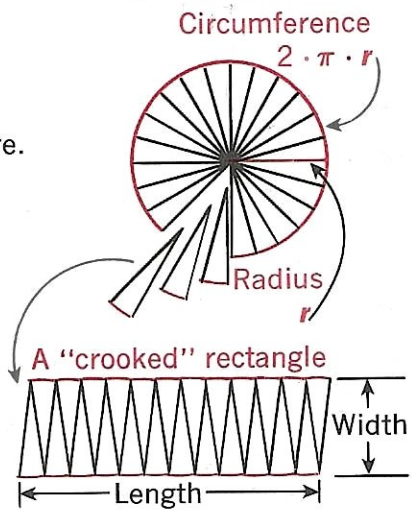
★ **9.** The outside diameter of a small auto tire is 25 in. How many revolutions does this wheel make in 1 mi? How many revolutions, at 60 mph, in 1 min?

12-7 AREAS OF CIRCULAR REGIONS



The circle in the figure has a radius of 10 units. Count squares in the shaded part of the figure and multiply by 4 to get an estimate of the area of the circle. If you do this carefully you will get an area of about 312 unit squares. Notice that this is very close to $\pi \cdot r \cdot r$ (r = radius).

To see that the area of a circle is indeed $\pi \cdot r^2$, think of breaking the circle into many small wedges as in the figure. Then think of putting the wedges back together to form a "crooked" rectangle.



Discussion Exercises

1. How does the area of the crooked rectangle compare with the area of the circle?

2. Explain why the width of the rectangle is about r.

3. Explain why the length of the rectangle is about $\pi \cdot r$.

4. What formula will give a good approximation for the area of the rectangle?

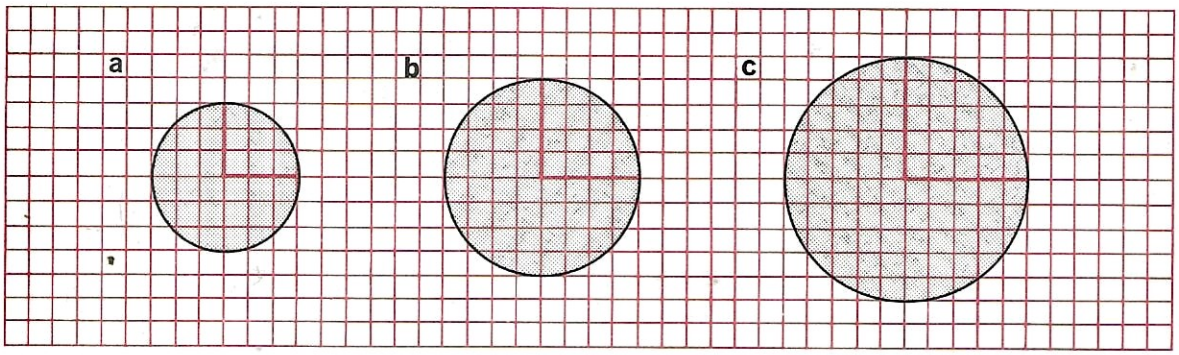
5. The formula for the area of a circle with radius r is

$$A = \pi \cdot r^2$$

Explain this formula, using your work in Exercises 1 through 4.

Exercises

1. Count squares or parts of squares to estimate the area of each circular region. The heavier colored lines are given so that you can count the squares in $\frac{1}{4}$ of each circle.



2. The radii for the circles in Exercise 1 are 3, 4, and 5 units. Using 3.14 for π, compute areas for each circle and check these with your estimates.

3. Find the area of each circular region with radius (r) or diameter (d) as indicated. Use $\pi \approx 3.14$.

[a] $r = 1$ ft
[b] $r = 10$ in.
[c] $r = 6$ cm
[d] $r = 2.3$ ft
[e] $r = 24.1$ m
[f] $d = 10$ yd
[g] $d = 2$ mi
[h] $d = 5$ cm

4. Find the area of each circular region with radius (r) or diameter (d) as indicated. Use $\pi \approx \frac{22}{7}$.

[a] $r = 1$ ft
[b] $r = 7$ in.
[c] $r = 10$ cm
[d] $r = \frac{7}{10}$ ft
[e] $r = \frac{7}{22}$ in.
[f] $d = \frac{7}{11}$ yd
[g] $d = 2\frac{1}{2}$ mi
[h] $d = 9$ cm

5. For these exercises, use $\pi \approx \frac{22}{7}$.

[a] The area of a circle is 22 sq ft. What is r^2?

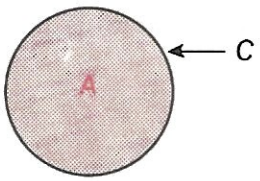
[b] The area of a circle is $\frac{88}{7}$ sq in. What is its radius?

[c] The area of a circle is $28\frac{2}{7}$ sq cm. What is its radius?

★[d] The area of a circle is $\frac{352}{700}$ sq in. What is its diameter?

★[e] The area of a circle is $\frac{220}{7}$ sq ft. What is its circumference?

6. The area of a circle is equal to its circumference. What is its radius?



$C = A$

7. Copy and complete the table, using $\pi \approx 3.14$.

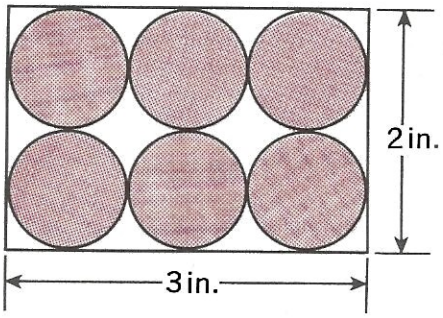
	Radius (r)	Diameter (d)	Circumference (C)	Area (A)
[a]	12 in.			
[b]		8 cm		
[c]			314 in.	
[d]				314 sq in.

8. A circle whose diameter is 4 in. is cut out of a square piece of tin 4 in. on a side. What is the area of the piece of tin which is left over?

★ **9.** A circular flower bed is surrounded by a walk 6 ft wide. The diameter of the flower bed is 60 ft. What is the surface area of this walk?

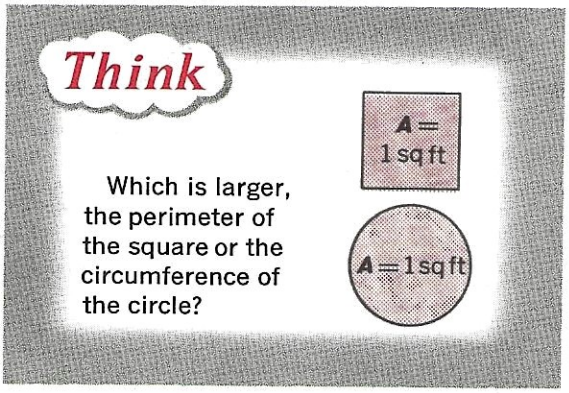
★**10.** The cost of building the walk of Exercise 8 is \$8.50 per square yard. What is the total cost of the walk?

★**11.** In the figure, six circular disks, each of which has a diameter of 1 in., are placed on a rectangular region 3 by 2 in. What percent of the surface of the rectangle is covered by the circles?



★**12.** [a] If the rectangle of Exercise 11 is covered by circular disks of .1-in. diameter, placed in a similar pattern, what percent is covered?

[b] What percent of the rectangle would be covered by disks whose diameters are .01 in.?



CHAPTER REVIEW

Key Concepts

1. It is important to be able to identify chords, arcs, inscribed angles, and central angles in a circle.
2. An **inscribed polygon** has all its vertices on the circle.
3. A **circumscribed polygon** has all its sides tangent to the circle.
4. A line is **tangent** to a circle if the line and circle intersect in exactly one point.
5. In a circle, a radius to the point of tangency is perpendicular to the tangent.
6. **Polyhedrons** have faces made up of plane polygonal regions.
7. The **circumference** of a circle is the distance around the circle.
8. The number $\frac{C}{d}$ is the same for all circles and is named by the Greek letter π.
9. The circumference of a circle is given by the formula $C = \pi \cdot d = 2 \cdot \pi \cdot r$.
10. The area of a circle is given by the formula $A = \pi \cdot r^2$.
11. Useful approximations for π are 3.14 and $\frac{22}{7}$.

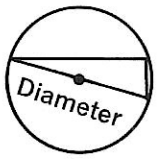
Vocabulary List

arc	circumscribed polygon	pi	semicircle
central angle	diameter	point of tangency	subtend
chord	inscribed angle	polyhedron	tangent
circumference	inscribed polygon	regular polygon	

Review Exercises

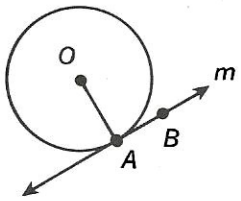
1. Use only one drawing to show your answers.
   [a] Draw a circle with a 2-in. radius and a center at O.
   [b] Draw a diameter and name it $\overline{AB}$.
   [c] Draw a chord $\overline{AC}$.
   [d] Draw an inscribed angle that subtends $\overset{\frown}{AC}$.
   [e] Draw the central angle for $\overset{\frown}{BC}$.

2. An angle that subtends a diameter is what kind of angle?



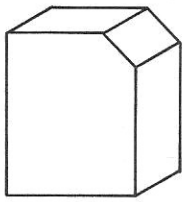
3. Draw two circles.
   [a] Draw an inscribed simple quadrilateral in one circle.
   [b] Sketch a circumscribed simple pentagon about the second circle.

4. Line m is tangent to the circle O at point A. What can you say about $\angle OAB$?

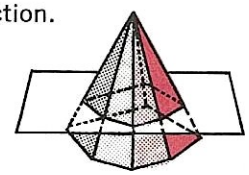


5. Draw a line and mark a point R on the line. Construct two circles, one on each side of the line, that are tangent to the line.

6. Look at the polyhedron.
   [a] How many vertices?
   [b] How many faces?
   [c] How many edges?
   [d] What is $(V + F) - E$ for this polyhedron?



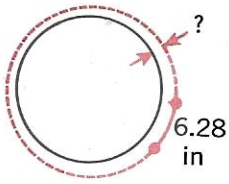
7. Sketch this cross section.



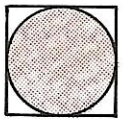
8. What is the circumference of a circle with a diameter of 7 in.? Use $\pi \approx \frac{22}{7}$.
9. What is the area of a circular region with a radius of $3\frac{1}{2}$ in.? Use $\pi \approx \frac{22}{7}$.
10. What is the circumference of a circle with a radius of 6.2? Use $\pi \approx 3.14$.
11. What is the area of a circle with a diameter 7.4? Use $\pi \approx 3.14$.
12. If the area of a circle is $\frac{22}{7}$, what is the radius? Use $\pi \approx \frac{22}{7}$.
13. If the circumference of a circle is 31.4 in., what is the diameter? Use $\pi \approx 3.14$.

★14. Johnny Workhorse found the circumference and the area of 30 circles. In each case he found that, very nearly, the area divided by the circumference was $\frac{1}{4}$ the diameter. Is this really true? Why?

★15. A circular iron rim is cut and a 6.28-in. piece is welded into the rim. How much bigger is the radius of the rim?



★16. What percent of the square is filled by the disc?



CHAPTER SUPPLEMENT

Reading List

ADLER, IRVING. *The Giant Golden Book of Mathematics.* New York, Golden Press, 1960; pp. 26–27, 50.

DAVIS, PHILIP J. *The Lore of Large Numbers.* New York, Random House, 1961; pp. 55–75.

GARDNER, MARTIN. *The Scientific Book of Puzzles and Diversions.* New York, Simon and Schuster, 1959; pp. 73–83.

———. *The Second Scientific American Book of Mathematical Puzzles and Diversions. Ibid.*, 1961; pp. 51–64, 89–103.

JOHNSON, DONOVAN A. *Curves in Space.* Exploring Mathematics on Your Own; St. Louis, Webster Publications, 1963; pp. 16–22.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION. *What's Up There.* Washington, D.C., U.S. Government Printing Office, 1964; entire book.

RAVIELLI, ANTHONY. *An Adventure in Geometry.* New York, Viking Press, 1957; pp. 58–81.

RUCHLIS, HY, and ENGELHARDT, JACK. *The Story of Mathematics—Geometry for the Young Scientist.* Irvington-on-Hudson, New York, Harvey House Publishing, 1958; pp. 34-71.

Activities

A. Read about "phi, the golden ratio," and report your findings to your class. (See *The Second Scientific American Book of Mathematical Puzzles and Diversions,* pp. 89–103.)

B. Find four circular objects in your home. Measure the diameter of each to the nearest $\frac{1}{4}$ inch. Then figure out the circumference of each.

C. Using a straightedge and compass, construct regular polygons of 3, 4, 6, 8, and 12 sides each.

D. With colored paper, construct four different polyhedrons. Use them to decorate your classroom.

E. Read about the game of "Hex." Study the rules and strategy and then challenge a classmate to play. (See *The Scientific American Book of Mathematical Puzzles and Diversions,* pp. 73–83.)

F. Figure out some designs (similar to those shown in Exercise 8, page 382) which you can do with ruler and compass. See if a classmate can copy your designs.

CUMULATIVE REVIEW

1. Find two numbers a and b so that $a \cdot b = 27$ and $a \div b = 3$.

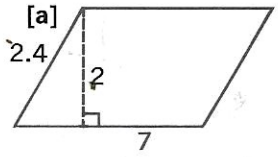
2. Solve the equations

[a] $a + 4\frac{2}{5} = 9\frac{3}{5}$ [c] $(3 \cdot c) + 4 = 31$

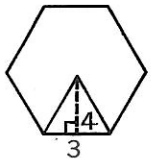
[b] $\frac{2}{3} \cdot d = 24$ [d] $4.62 + b = 11.84$

3. Find the area and perimeter of each polygon.

[a]

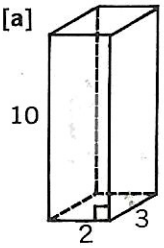


[b]

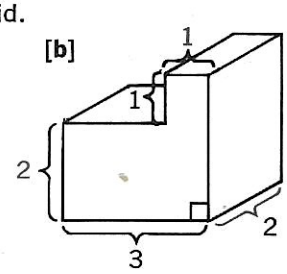


4. Find the surface area and volume of each solid.

[a]



[b]



5. Give the prime factorizations of

[a] 75 [b] 80 [c] 56 [d] 147

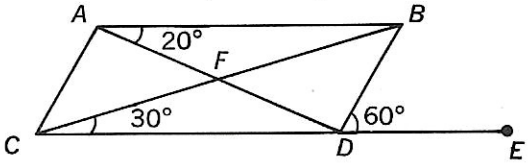
6. Reduce to lowest terms.

[a] $\frac{18}{24}$ [b] $\frac{42}{30}$ [c] $\frac{54}{81}$ [d] $\frac{56}{147}$

7. Give the correct sign ($<$, $>$, or $=$) for each ▒.

[a] $\frac{1}{4}$ ▒ $\frac{1}{6}$ [b] $\frac{3}{4}$ ▒ $\frac{5}{6}$ [c] $\frac{42}{54}$ ▒ $\frac{7}{9}$

8. Figure $ABCD$ is a parallelogram.



Determine the number of degrees in each of the following angles.

[a] $\angle ABC$ [c] $\angle ADB$

[b] $\angle ADC$ [d] $\angle BFD$

9. Which number is the GCF of 20 and 36?

[a] 2 [b] 180 [c] 4 [d] 12

10. Give the reciprocal for

[a] 6 [b] $\frac{1}{8}$ [c] 3.4 [d] 100

11. Copy and complete the table.

[a]

function rule $n \div 2\frac{1}{2}$	
n	$f(n)$
5	2
3	▒
▒	4
▒	$2\frac{2}{5}$
0	▒

[b]

function rule ▒	
n	$f(n)$
3	9
0	3
$\frac{1}{2}$	4
$3\frac{1}{2}$	▒
▒	15

12. Solve the equations.

[a] $(3 \cdot r) + 7 = 28$ [b] $4\frac{2}{3} \div n = 14$

13. One photograph is 2 by 3 in. If the width and length of another photograph are each double the first, what is the ratio of their areas?

14. Give the LCM of 7 and 11; 30 and 21.

15. Show points A, B, and C on a line such that $\overline{BA} < \overline{AC} < \overline{BC}$.

The following discussion refers to questions 16 through 22: A scientist was comparing the weights of five differently colored stones. He collected the following data.

Color	Weight (oz)
Red	13.42
Blue	7.006
Yellow	11.1
Green	7.25
Pink	9.3264

16. What are the combined weights of the green and red stones?

17. Which is heavier, the blue stone or the green one? By how much?

18. If the yellow stone were four times heavier than it is now, how much would it weigh?

19. If the green stone were split exactly in half, how much would each half weigh?

20. What is the product of the weights of the blue and yellow stones?

21. The pink stone is how many times as heavy as the green stone? (Give your answer to the ten-thousandths place.)

22. Write the weight of the blue stone in the form of a fraction.

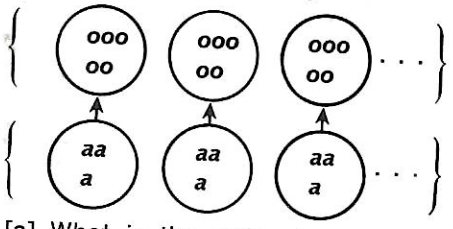
23. The Gold team won 17 games out of 25. The Silver team won 13 games out of 20.

[a] What percent of its games did the Gold team win?

[b] What percent of its games did the Silver team win?

[c] What team had the better record?

24. Look at the following diagram.



[a] What is the ratio of oranges (**o**) to apples (**a**)?

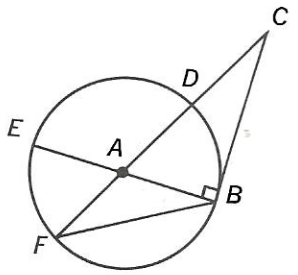
[b] How many oranges for 12 apples?

[c] How many apples for 60 oranges?

25. A bookstore is having a sale giving 20% off on all books. If a book was originally marked \$2.15, what is its sale price?

26. A board is 12 in. long. If its length is to be decreased by 75%, what will the new length be?

27. Point A is the center of the circle.



[a] Name a radius of the circle.

[b] Name a diameter of the circle.

[c] Name a tangent line to the circle.

[d] Name an inscribed angle of the circle.

[e] Name a central angle of the circle.

28. In the figure for Exercise 27, what kind of a triangle is

[a] $\triangle ABC$? [b] $\triangle ABF$?

29. A circular field is 50 ft in diameter. What would be the length of a fence that completely surrounded the edge of the field? (Use $\pi = 3.14$.)

30. In Exercise 29, what would be the area of a ground cloth used to cover the field? (Use $\pi = 3.14$.)

31. The circumference of a circle is $37\frac{5}{7}$ in. Using $\pi = \frac{22}{7}$, compute its area.

Use a ruler and compass to complete the constructions for Exercises 32 and 33.

32. Inscribe a hexagon in a circle of radius 2 in.

33. Inscribe a triangle in a circle of radius 3 in.

"Take some more tea," the March Hare said to Alice, very earnestly.

"I've had nothing yet," Alice replied in an offended tone: "so I can't take more."

"You mean you can't take less," *said the Hatter: "it's very easy to take* more *than nothing."*

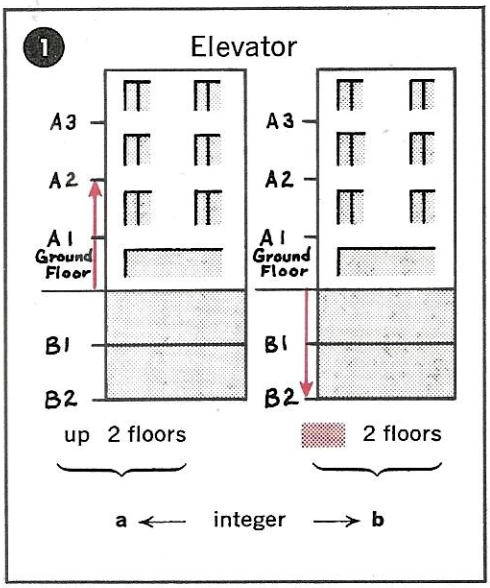
13

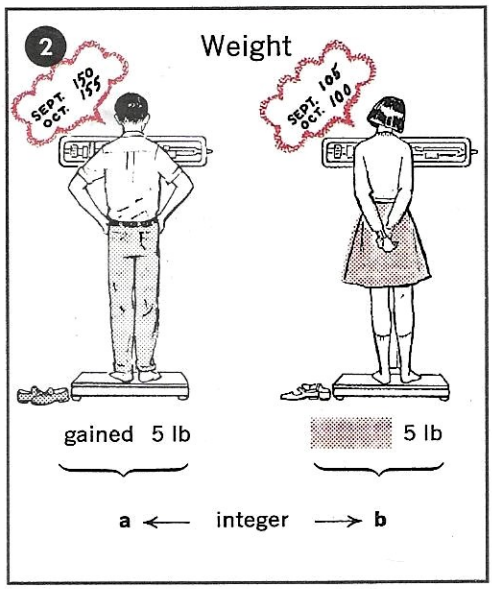
Integers

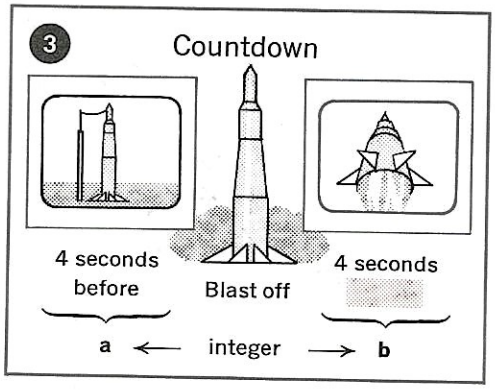
13-1 THE SET OF INTEGERS

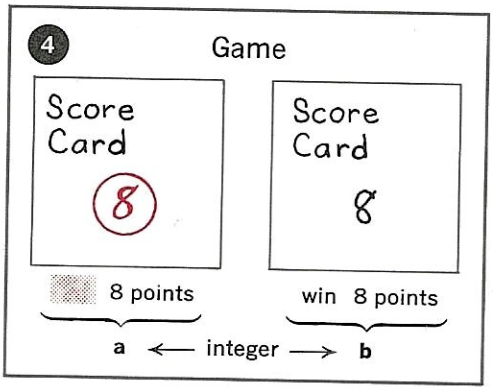
Discussion Exercise

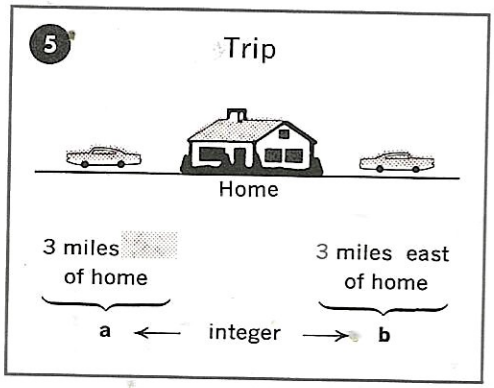
Study Examples 1 through 8 and tell what word you think should be written in each screen.

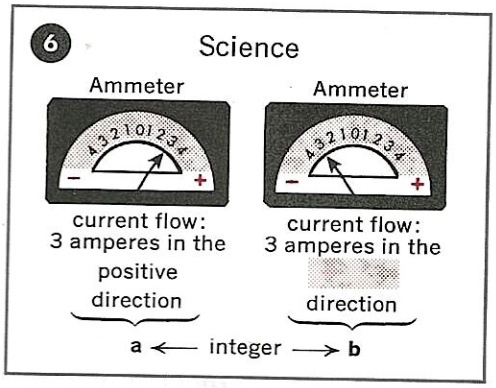


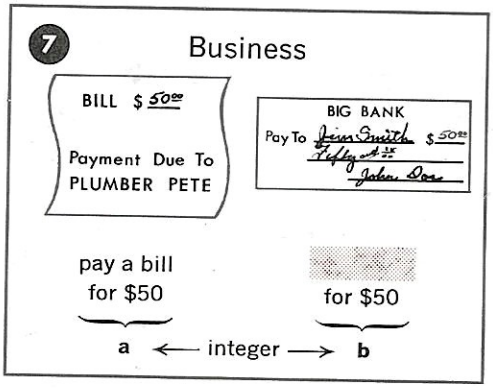


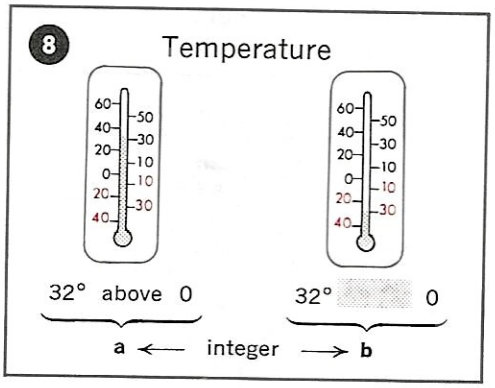




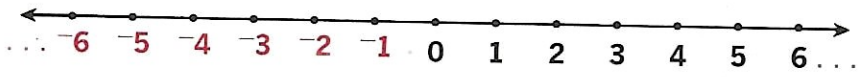








The pictures above suggest a need to invent numbers for the points opposite the whole number points on the number line. The symbols for some of these numbers are given in color on the number line below. These numbers are called negative integers.



For each nonzero whole number **(positive integer)** we want a matching negative integer. Study the first four rows of the table.

Positive and **negative** integers, together with **zero**, form a set of numbers called **the integers**.

Negative integers	Positive integers
$^{-}1, ^{-}2, ^{-}3, ^{-}4, \ldots$	0 $1, 2, 3, 4, \ldots$

Negative integer ← → Positive integer		
$^{-}1$	matches with	1
$^{-}2$	matches with	2
$^{-}3$	matches with	3
$^{-}4$	matches with	4
a	matches with	5
⋮		⋮
b	matches with	89
⋮		⋮
$^{-}123$	matches with	**c**
⋮		⋮
d	matches with	5647
⋮		⋮

The set of integers = $\{0,\ 1, ^{-}1,\ 2, ^{-}2,\ 3, ^{-}3,\ 4, ^{-}4,\ 5, ^{-}5,\ \ldots\}$

Exercises

1. For each Example 1 through 8 (pages 403–404), give the integers that could be used to describe the situation. Example: 8a, 32; 8b, $^{-}32$

2. Give the integers for Parts **a** through **d** of the above table.

3. Pairs of integers, such as 3 and $^{-}3$ are sometimes called **opposites**. We could say

$^{-}3$ is the opposite of 3, or 3 is the opposite of $^{-}3$.

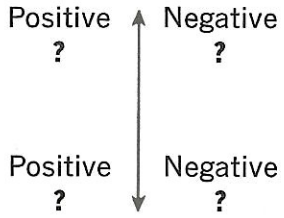
Give the opposite of each integer.

[a] 5	[d] $^{-}4$	[g] 25	[j] 586	[m] $^{-}5936$
[b] 8	[e] $^{-}9$	[h] 59	[k] $^{-}1079$	[n] 100
[c] 10	[f] $^{-}11$	[i] $^{-}86$	[l] 1,687,943	[o] $^{-}863{,}650$

4. Write the integers along the number line from $^{-}6$ to $^{-}20$.

★**5.** [a] What is the opposite of the opposite of 8?
[b] What is the opposite of the opposite of the opposite of 8?

★**6.** You may wonder why the whole numbers "go to the right" and the negative numbers "go to the left." It is simply a matter of choice.



[a] Draw a 6-in. horizontal picture of the number line. Choose a zero point. Choose the positive number direction to the left. Mark a point every $\frac{1}{2}$ in. Label the number line.

[b] Draw a 6-in. vertical picture of the number line. Choose a zero point. Choose a positive number direction either up or down and mark points every $\frac{1}{2}$ in. Label your number line.

Think

Find the numerator a, if $\frac{1}{15} < \frac{a}{100}$ and $\frac{1}{14} > \frac{a}{100}$.

13-2 BASIC PRINCIPLES FOR ADDITION OF INTEGERS

Mathematics is a creation of man's mind. A mathematician might invent numbers as a writer invents characters in a play. As an inventor of numbers, he chooses **names** for numbers and decides what marks he will use to represent them. Also, it is necessary that he say what principles hold true for the numbers.

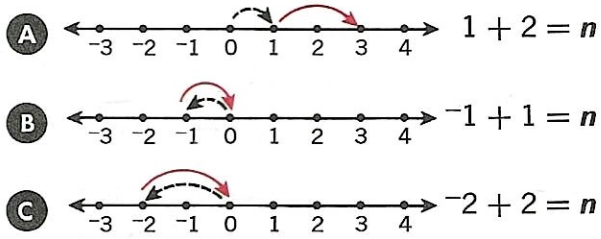
In Section 13–1, we named the integers and agreed upon symbols to represent them. We will now consider some basic principles for the integers. The number line suggests an important principle for addition of integers.

Discussion Exercises

1. Figure A reviews whole number addition on the number line. Explain each jump and give the sum.

2. Figure B suggests a way to think about the sum $^{-}1 + 1$. Explain each jump and give the sum.

3. Explain each jump in Figure C and give the sum.



4. Draw a number line and explain how to use "jumps" on the number line to find each sum.

[a] $^{-}3 + 3 = n$ [b] $^{-}4 + 4 = n$ [c] $^{-}6 + 6 = n$ [d] $^{-}10 + 10 = n$

We have seen that $^{-}1 + 1 = 0$, $^{-}2 + 2 = 0$, and so on. These results suggest that we should agree to accept the first basic principle for addition of integers listed in the table which follows. The opposites principle and the other principles will help you figure out how to find the sum of any two integers. An important result of the commutative and associative principles is described in the statement below the table.

Basic principles for addition of integers

Opposites principle

The sum of each negative integer and its opposite is zero.

Examples: $^{-}1 + 1 = 0$ and $^{-}2 + 2 = 0$

Zero principle

If we choose an integer and add zero, the sum is the integer we chose.

Examples: $^{-}8 + 0 = {}^{-}8$ and $^{-}15 + 0 = {}^{-}15$

In general, for any integer a, $a + 0 = a$.

Commutative principle

When finding the sum of two integers, we can change the order of the addends and the sum is the same.

Examples:

$^{-}4 + {}^{-}3 = {}^{-}3 + {}^{-}4$

$^{-}5 + 5 = 5 + {}^{-}5$

In general, for integers a and b,

$a + b = b + a$

Associative principle

When finding the sum of three integers, we can change the grouping of the addends and the sum is the same.

Examples:

$(^{-}2 + {}^{-}5) + 5 = {}^{-}2 + (^{-}5 + 5)$

$(8 + 7) + {}^{-}7 = 8 + (7 + {}^{-}7)$

In general, for integers a, b, and c,

$(a + b) + c = a + (b + c)$

Since we can change both the **order** and the **grouping** of the addends, we can rearrange addends in any way that is convenient.

Exercises

1. Find the sums.

[a] $^{-}5 + 5$ [c] $^{-}23 + 23$ [e] $^{-}3 + 0$ [g] $^{-}18 + 0$

[b] $^{-}8 + 8$ [d] $^{-}967 + 967$ [f] $^{-}9 + 0$ [h] $^{-}76 + 0$

2. Find the integer for a. Then find the integer for b. The opposites and zero principles will help you complete these exercises.

[a] $^{-}1 + 1 = a \longrightarrow 1 + {}^{-}1 = b$

[b] $^{-}7 + 7 = a \longrightarrow 7 + {}^{-}7 = b$

[c] $^{-}56 + 56 = a \longrightarrow 56 + {}^{-}56 = b$

[d] $^{-}8 + 0 = a \longrightarrow 0 + {}^{-}8 = b$

[e] $^{-}72 + 0 = a \longrightarrow 0 + {}^{-}72 = b$

[f] $^{-}963 + 0 = a \longrightarrow 0 + {}^{-}963 = b$

3. Find the missing integer.

[a] ▒ is the integer that adds to 8 to give the sum 0.

[b] ▒ is the integer that adds to $^{-}17$ to give the sum 0.

[c] ▒ is the integer that adds to 58 to give the sum 0.

[d] ▒ is the integer that adds to $^{-}867$ to give the sum 0.

4. Find the integer for a. Then find the integer for b.

[a] $2 + (3 + {}^{-}3) = a \longrightarrow (2 + 3) + {}^{-}3 = b$

[b] $6 + (4 + {}^{-}4) = a \longrightarrow (6 + 4) + {}^{-}4 = b$

[c] $^{-}3 + (^{-}5 + 5) = a \longrightarrow (^{-}3 + {}^{-}5) + 5 = b$

[d] $^{-}26 + (^{-}8 + 8) = a \longrightarrow (^{-}26 + {}^{-}8) + 8 = b$

5. Solve the equations. The letters represent integers.

[a] $^{-}4 + x = {}^{-}4$ [d] $t + {}^{-}12 = 0$ [g] $n + (^{-}5 + {}^{-}8) = (6 + {}^{-}5) + {}^{-}8$

[b] $7 + n = 0$ [e] $9 + {}^{-}9 = n + 9$ [h] $^{-}13 + (t + 4) = (^{-}13 + {}^{-}8) + 4$

[c] $r + 0 = {}^{-}6$ [f] $^{-}56 + k = 0 + {}^{-}56$ [i] $(^{-}72 + 4) + {}^{-}45 = {}^{-}72 + (4 + s)$

6. Find the integer for a. Then find the integer for b. Remember that the commutative and associative principles together suggest that you can rearrange addends in any way you choose.

[a] Since $(4 + {}^{-}4) + (5 + {}^{-}5) = a$, then $(4 + 5) + (^{-}4 + {}^{-}5) = b$.

[b] Since $(6 + {}^{-}6) + {}^{-}8 = a$, then $(6 + {}^{-}8) + {}^{-}6 = b$.

[c] Since $17 + (9 + {}^{-}9) = a$, then $(^{-}9 + 17) + 9 = b$.

★**7.** Solve the equations.

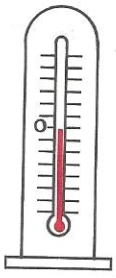
[a] $(6 + {}^{-}7) + {}^{-}6 = n$ [c] $(n + {}^{-}8) + {}^{-}4 = {}^{-}8$ [e] $(n + {}^{-}2) + {}^{-}4 = 0$

[b] $3 + (^{-}7 + 4) = n$ [d] $(^{-}7 + n) + 7 = 15$ [f] $5 + (^{-}3 + {}^{-}2) = n$

13-3 ADDITION OF INTEGERS

The table shows two ways to think about addition of integers.

Addition of integers

Using mathematical principles	Using physical examples
A Since $5 + (3 + {}^{-}3) = \blacksquare$, then $8 + {}^{-}3 = \blacksquare$.	**B** Receive a check for \$8.00. Pay a bill for \$3.00. Richer or poorer? How much richer or poorer? $8 + {}^{-}3 = \blacksquare$
C Since $(7 + {}^{-}3) + {}^{-}4 = \blacksquare$, then ${}^{-}3 + {}^{-}4 = \blacksquare$.	**D** Negative direction / Positive direction. Jump 3 units left. Jump 4 more units left. End up to left or right of starting point? How far? ${}^{-}3 + {}^{-}4 = \blacksquare$
E Since ${}^{-}3 + ({}^{-}4 + 4) = \blacksquare$, then ${}^{-}7 + 4 = \blacksquare$.	**F** From 0°, the temperature fell 7°. Then it rose 4°. Above or below 0°? How far? ${}^{-}7 + 4 = \blacksquare$ 

Discussion Exercises

1. Use Example A.
   [a] Give the missing numbers.
   [b] Prove that the sums you found are correct, using the opposites, the zero, and the associative principles.
2. Use Example B.
   [a] Find the sum.
   [b] Explain your reasoning.

3. Use Example C. Think of 7 as $4 + 3$.
[a] What number goes in the first ▒? What basic principles assure you that this is a correct answer?
[b] Since $^{-}3 + {}^{-}4$ adds to 7 to give 0, what number is the sum under the colored screen?

4. Use Example D.
[a] Find the sum. [b] Explain your reasoning.

5. Use Example E.
[a] Give the missing numbers.
[b] Use the basic principles to prove that the sums you found are correct.

6. Use Example F.
[a] Find the sum. [b] Explain your reasoning.

Exercises

As you continue to study mathematics, you will have many more opportunities to use basic principles and careful reasoning to draw conclusions. In Exercises 1 through 5 below, you are asked to find the sums, using basic principles rather than physical examples. Exercise 6 will remind you how physical examples can make the sums of integers seem reasonable.

1. Solve the equation to find the sum of the addends in the colored screen. Because of the basic principles, you can add any two addends first.

[a] $4 + {}^{-}3$
$1 + 3 + {}^{-}3 = n$

[b] $8 + {}^{-}6$
$2 + 6 + {}^{-}6 = n$

[c] $5 + {}^{-}2$
$3 + 2 + {}^{-}2 = n$

[d] $^{-}3 + 7$
$^{-}3 + 3 + 4 = n$

[e] $^{-}9 + 15$
$^{-}9 + 9 + 6 = n$

[f] $28 + {}^{-}12$
$16 + 12 + {}^{-}12 = n$

2. When you find the number for n in the cloud, you have found the sum of the two integers in the gray screen. Copy the equation, giving the correct sum.

[a] $n + 3$
$5 + {}^{-}3 = n$

[b] $n + 1$
$9 + {}^{-}1 = n$

[c] $n + 8$
$12 + {}^{-}8 = n$

[d] $n + 7$
$10 + {}^{-}7 = n$

[e] $4 + n$
$^{-}4 + 6 = n$

[f] $5 + n$
$^{-}5 + 9 = n$

[g] $7 + n$
$18 + {}^{-}7 = n$

[h] $14 + n$
$^{-}14 + 36 = n$

3. Find the sums.

3 + 2

[a] Since $5 + (^{-}3 + {}^{-}2) = 0$, then $^{-}3 + {}^{-}2 =$ ▒.

This number adds to 5 to give 0.

[b] Since $8 + (^{-}6 + {}^{-}2) = 0$, then $^{-}6 + {}^{-}2 =$ ▒.

[c] Since $7 + (^{-}4 + {}^{-}3) = 0$, then $^{-}4 + {}^{-}3 =$ ▒.

[d] Since $12 + (^{-}7 + {}^{-}5) = 0$, then $^{-}7 + {}^{-}5 =$ ▒.

[e] Since $25 + (^{-}20 + {}^{-}5) = 0$, then $^{-}20 + {}^{-}5 =$ ▒.

4. Solve the equations to find the sum of the addends in the colored screen.

[a] $^{-}5 + 2$

$^{-}3 + {}^{-}2 + 2 = n$

[b] $^{-}9 + 3$

$^{-}6 + {}^{-}3 + 3 = n$

[c] $^{-}4 + 1$

$^{-}3 + {}^{-}1 + 1 = n$

[d] $^{-}10 + 6$

$^{-}4 + {}^{-}6 + 6 = n$

[e] $5 + {}^{-}7$

$5 + {}^{-}5 + {}^{-}2 = n$

[f] $8 + {}^{-}18$

$8 + {}^{-}8 + {}^{-}10 = n$

5. When you find the number for n in the cloud, you have found the sum of the two integers in the gray screen. Copy the equation, giving the correct sum.

[a] $n + {}^{-}2$: $^{-}6 + 2 = n$

[b] $n + {}^{-}8$: $^{-}9 + 8 = n$

[c] $n + {}^{-}8$: $^{-}12 + 8 = n$

[d] $n + {}^{-}6$: $^{-}20 + 6 = n$

[e] $^{-}7 + n$: $7 + {}^{-}10 = n$

[f] $^{-}6 + n$: $6 + {}^{-}8 = n$

[g] $^{-}2 + n$: $2 + {}^{-}7 = n$

[h] $n + {}^{-}38$: $^{-}42 + 38 = n$

For Exercises 6 through 8, think about the physical examples. Then copy the equation, giving the correct sum.

6. Think about checks and bills.

[a] Receive a check for \$5. / Pay a bill for \$3. / How much richer or poorer?

$5 + {}^{-}3 = n$

[b] Pay a bill for \$3. / Pay a bill for \$6. / How much richer or poorer?

$^{-}3 + {}^{-}6 = n$

[c] Pay a bill for \$7. / Receive a check for \$5. / How much richer or poorer?

$^{-}7 + 5 = n$

7. Think about jumps on the number line, starting at zero.

	Jump 4 units right.	Jump 2 units left.	How far left or right of zero?	Jump 5 units left.	Jump 3 more units left	How far left or right of zero?	Jump 8 units left.	Jump 5 units right.	How far left or right of zero?
[a]	4	+ $^{-}2$	= n						
[b]				$^{-}5$	+ $^{-}3$	= n			
[c]							$^{-}8$	+ 5	= n

8. Think about a thermometer.

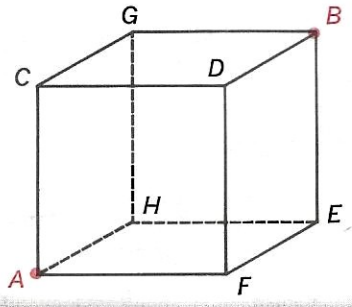
	35° above 0°	Temperature falls 5°.	New temperature?	12° below 0°.	Temperature falls 7°.	New temperature?	10° above 0°.	Temperature falls 15°.	New temperature?
[a]	35	+ $^{-}5$	= n						
[b]				$^{-}12$	+ $^{-}7$	= n			
[c]							10	+ $^{-}15$	= n

9. Find the sums. Use any method you choose.

[a] $3 + {}^{-}3$
[b] $10 + {}^{-}7$
[c] $12 + {}^{-}6$
[d] $12 + {}^{-}4$
[e] $5 + {}^{-}3$
[f] $^{-}7 + 9$
[g] $21 + {}^{-}11$
[h] $^{-}2 + {}^{-}3$
[i] $^{-}10 + {}^{-}5$
[j] $^{-}7 + {}^{-}3$
[k] $8 + {}^{-}2 + {}^{-}8$
[l] $^{-}2 + 1$
[m] $^{-}7 + 5$
[n] $^{-}4 + 2$
[o] $7 + {}^{-}8$
[p] $10 + {}^{-}17$
[q] $({}^{-}8 + {}^{-}5) + {}^{-}7$
[r] $({}^{-}4 + {}^{-}6) + 7$
[s] $5 + ({}^{-}8 + {}^{-}2)$
[t] $^{-}6 + (1 + 4)$
[u] $^{-}8 + 2 + {}^{-}3$
[v] $6 + {}^{-}3 + {}^{-}10$
[w] $85 + {}^{-}37$
[x] $82 + {}^{-}75$
[y] $71 + {}^{-}49$
[z] $1000 + {}^{-}712$

Think

The symbol *ACDB* describes a "trip" from *A* to *B* along just three edges of the cube. We agree that in a trip no edge is to be traversed twice.

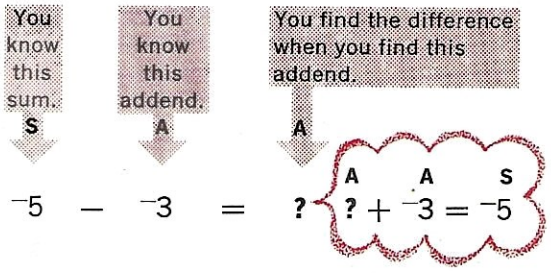


Tell how many different trips there are from *A* to *B* along just

1. three edges of the cube
2. four edges of the cube
3. five edges of the cube
4. seven edges of the cube

13-4 SUBTRACTION OF INTEGERS

As with subtraction of whole numbers, you can think of subtraction of integers as finding the missing addend.



Discussion Exercises

1. Find the difference in the example above. Explain your reasoning.

2. In each column, the difference of the two integers is given for each equation. Check the answers by adding the two addends to see if you get the sum. Do you see any patterns? Explain your answer.

S A A	S A A	S A A	S A A
[a] 5 − 2 = 3	[b] 3 − 1 = 2	[c] 0 − −2 = 2	[d] 0 − −2 = 2
5 − 1 = 4	2 − 1 = 1	1 − −2 = 3	−1 − −2 = 1
5 − 0 = 5	1 − 1 = 0	2 − −2 = 4	−2 − −2 = 0
5 − −1 = 6	0 − 1 = −1	3 − −2 = 5	−3 − −2 = −1
5 − −2 = 7	−1 − 1 = −2	4 − −2 = 6	−4 − −2 = −2
5 − −3 = 8	−2 − 1 = −3	5 − −2 = 7	−5 − −2 = −3

Exercises

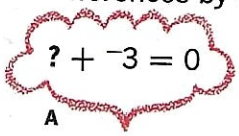
1. Find the sums.

[a] −8 + 3	[f] 0 + −4	[k] −3 + 2	[p] 16 + −10 + 4
[b] 5 + −6	[g] 1 + −1	[l] −6 + 9	[q] −5 + 5 + −5
[c] −6 + 8	[h] 16 + −8	[m] −2 + −2 + −2	[r] 6 + −2 + 8
[d] −4 + −3	[i] −10 + −1	[n] −3 + 4 + −6	[s] −4 + −5 + −6
[e] −9 + 19	[j] −10 + 1	[o] −8 + 4 + 4	[t] −8 + 2 + 8

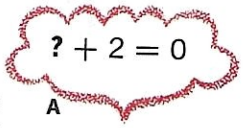
2. Find the differences by finding the missing addends.

A A S	A A S	A A S	S A A
[a] n + −1 = −5	[f] n + 6 = 3	[k] n + −3 = −5	[p] 12 = n + −3
[b] n + −3 = −4	[g] −2 + n = 5	[l] 8 + n = −2	[q] 8 = −6 + n
[c] n + 7 = 0	[h] n + −4 = 2	[m] −3 + n = −7	[r] −3 = n + 2
[d] 5 + n = 3	[i] n + −3 = 5	[n] n + −6 = −10	[s] −7 = −4 + n
[e] n + 2 = 1	[j] n + 4 = −1	[o] 9 + n = 2	[t] −9 = 9 + n

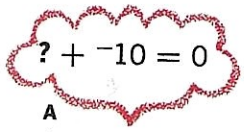
3. Find the differences by finding the missing addends.



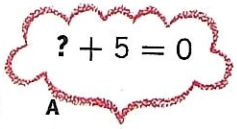
S A
[a] $0 - {}^{-}3 = A$



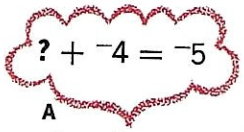
S A
[b] $0 - 2 = A$



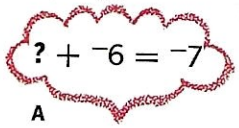
S A
[c] $0 - {}^{-}10 = A$



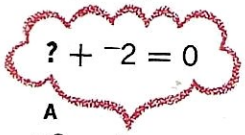
S A
[d] $0 - 5 = A$



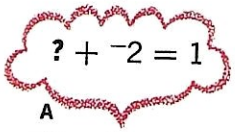
S A
[e] $^{-}5 - {}^{-}4 = A$



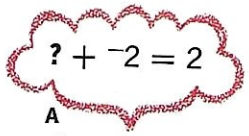
S A
[f] $^{-}7 - {}^{-}6 = A$



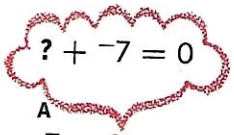
S A
[g] $0 - {}^{-}2 = A$



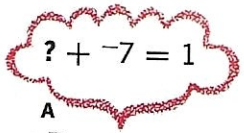
S A
[h] $1 - {}^{-}2 = A$



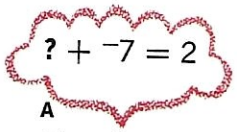
S A
[i] $2 - {}^{-}2 = A$



S A
[j] $0 - {}^{-}7 = A$



S A
[k] $1 - {}^{-}7 = A$



S A
[l] $2 - {}^{-}7 = A$

4. Find the differences.

S A
[a] $0 - 3 = A$
[b] $^{-}1 - 3 = A$
[c] $^{-}2 - 3 = A$
[d] $^{-}4 - 3 = A$

S A
[e] $0 - 9 = A$
[f] $^{-}1 - 9 = A$
[g] $^{-}2 - 9 = A$
[h] $^{-}3 - 9 = A$

S A
[i] $0 - {}^{-}1 = A$
[j] $1 - {}^{-}1 = A$
[k] $2 - {}^{-}1 = A$
[l] $8 - {}^{-}1 = A$

S A
[m] $^{-}2 - 3 = A$
[n] $^{-}4 - {}^{-}3 = A$
[o] $5 - {}^{-}1 = A$
[p] $^{-}5 - {}^{-}3 = A$

5. Find the differences.

[a] $\begin{array}{r} 5 \\ \underline{-3} \end{array}$ [b] $\begin{array}{r} 5 \\ \underline{-2} \end{array}$ [c] $\begin{array}{r} 5 \\ \underline{-1} \end{array}$ [d] $\begin{array}{r} 5 \\ \underline{-0} \end{array}$ [e] $\begin{array}{r} 5 \\ \underline{-{}^{-}1} \end{array}$ [f] $\begin{array}{r} 5 \\ \underline{-{}^{-}2} \end{array}$ [g] $\begin{array}{r} 5 \\ \underline{-{}^{-}3} \end{array}$

6. Find the integer for ***a***. Then find the integer for ***b***.

[a] $0 - {}^{-}3 = a \longrightarrow 1 - {}^{-}3 = b$
[b] $0 - {}^{-}3 = a \longrightarrow 2 - {}^{-}3 = b$
[c] $0 - {}^{-}3 = a \longrightarrow 3 - {}^{-}3 = b$
[d] $0 - {}^{-}3 = a \longrightarrow {}^{-}1 - {}^{-}3 = b$
[e] $0 - 4 = a \longrightarrow 1 - 4 = b$
[f] $0 - 4 = a \longrightarrow 2 - 4 = b$
[g] $0 - 4 = a \longrightarrow {}^{-}1 - 4 = b$
[h] $0 - {}^{-}5 = a \longrightarrow 4 - {}^{-}5 = b$
[i] $0 - 7 = a \longrightarrow 2 - 7 = b$
[j] $0 - 6 = a \longrightarrow {}^{-}1 - 6 = b$
[k] $0 - {}^{-}8 = a \longrightarrow {}^{-}1 - {}^{-}8 = b$
[l] $0 - {}^{-}6 = a \longrightarrow 3 - {}^{-}6 = b$
[m] $0 - {}^{-}7 = a \longrightarrow {}^{-}2 - {}^{-}7 = b$
[n] $0 - {}^{-}10 = a \longrightarrow 5 - {}^{-}10 = b$

7. Think about what the missing addend would be if the sum were zero. Then find the missing addend.

S A A

[a] $2 - {}^{-}4 = n$
[b] $3 - {}^{-}2 = n$
[c] ${}^{-}5 - {}^{-}2 = n$
[d] ${}^{-}4 - {}^{-}3 = n$
[e] ${}^{-}1 - {}^{-}1 = n$
[f] ${}^{-}4 - 5 = n$

S A A

[g] ${}^{-}4 - {}^{-}2 = n$
[h] ${}^{-}7 - {}^{-}4 = n$
[i] ${}^{-}2 - {}^{-}3 = n$
[j] ${}^{-}1 - {}^{-}5 = n$
[k] ${}^{-}8 - {}^{-}7 = n$
[l] ${}^{-}4 - {}^{-}3 = n$

S A A

[m] ${}^{-}5 - {}^{-}1 = n$
[n] ${}^{-}3 - {}^{-}2 = n$
[o] ${}^{-}8 - {}^{-}7 = n$
[p] ${}^{-}9 - {}^{-}4 = n$
[q] $14 - {}^{-}6 = n$
[r] $9 - {}^{-}5 = n$

S A A

[s] $1 - 3 = n$
[t] $4 - 9 = n$
[u] ${}^{-}13 - {}^{-}5 = n$
[v] ${}^{-}11 - {}^{-}4 = n$
[w] $74 - {}^{-}3 = n$
[x] ${}^{-}16 - {}^{-}14 = n$

8. Find the differences.

[a] ${}^{-}3 - 7$
[b] $3 - 7$
[c] ${}^{-}8 - 2$
[d] ${}^{-}8 - {}^{-}2$
[e] ${}^{-}2 - 8$
[f] $2 - 8$
[g] ${}^{-}2 - {}^{-}8$
[h] $8 - {}^{-}2$
[i] ${}^{-}5 - 3$
[j] $5 - {}^{-}3$
[k] ${}^{-}3 - {}^{-}8$
[l] $7 - {}^{-}3$
[m] $3 - 5$
[n] ${}^{-}6 - 3$
[o] ${}^{-}2 - {}^{-}1$
[p] ${}^{-}8 - 12$
[q] $15 - {}^{-}3$
[r] ${}^{-}6 - 9$
[s] ${}^{-}8 - 7$
[t] ${}^{-}2 - {}^{-}3$

9. Solve the equations. It may help you to think about sums and addends.

[a] $x - 3 = {}^{-}5$
[b] $x - {}^{-}3 = {}^{-}5$
[c] $3 - x = {}^{-}5$
[d] $3 - x = 5$
[e] $y = 3 - {}^{-}10$
[f] $y - 3 = {}^{-}10$
[g] $3 + y = {}^{-}10$
[h] $3 - y = {}^{-}10$
[i] $5 - x = 10$
[j] $y + y = {}^{-}8$
[k] ${}^{-}8 - y = y$
[l] ${}^{-}10 - y = y$
[m] ${}^{-}5 - c = 5$
[n] $c - 6 = {}^{-}6$
[o] $d - {}^{-}7 = {}^{-}7$
[p] ${}^{-}7 - d = 1$

Think

Ben argued that his $5.00 a week allowance was not enough. His father refused to raise the amount.

Ben offered the following "bargain." He asked his father to give him 1¢ for the first week, 2¢ for the second week, 4¢ for the following week, 8¢ the following week, and so on.

Should Ben's father accept the bargain? Explain your answer.

13-5 MULTIPLICATION AND DIVISION OF INTEGERS

The basic principles for multiplication of integers are suggested by the basic principles for multiplication of whole numbers. These principles, along with an important result of the commutative and associative principles, are shown in the following table.

Basic principles for multiplication of integers

One principle

If we choose an integer and multiply by 1, the product is the integer we chose.

Examples: $^{-}5 \cdot 1 = {}^{-}5$, $8 \cdot 1 = 8$, $^{-}64 \cdot 1 = {}^{-}64$

In general, for any integer a,

$$a \cdot 1 = a$$

Commutative principle

When finding the product of two integers, we can change the order of the factors and the product is the same.

Examples: $^{-}3 \cdot {}^{-}4 = {}^{-}4 \cdot {}^{-}3$

$^{-}5 \cdot 2 = 2 \cdot {}^{-}5$

In general, for integers a and b,

$$a \cdot b = b \cdot a$$

Associative principle

When finding the product of three integers, we can change the grouping of the factors and the product is the same.

Examples: $(^{-}2 \cdot {}^{-}5) \cdot 8 = {}^{-}2 \cdot ({}^{-}5 \cdot 8)$

$(5 \cdot {}^{-}4) \cdot {}^{-}3 = 5 \cdot ({}^{-}4 \cdot {}^{-}3)$

In general, for integers a, b, and c,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Since we can change both the order and the grouping of the factors, we can rearrange factors in any way that is convenient.

The **distributive principle** relates multiplication and addition to each other.

A basic principle for multiplication and addition

Distributive principle

Examples: $^{-}5 \cdot 2 = {}^{-}5 \cdot (1 + 1) = (^{-}5 \cdot 1) + (^{-}5 \cdot 1)$

$3 \cdot {}^{-}5 = (2 + 1) \cdot {}^{-}5 = (2 \cdot {}^{-}5) + (1 \cdot {}^{-}5)$

In general, for integers a, b, and c,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

As you continue to study mathematics, you will find many more mathematical ideas that cannot be easily illustrated using physical examples. It is difficult, for instance, to find a simple physical example to illustrate multiplication of integers. The following table shows one possibility. The exercises will help you understand how to use the basic principles to find products.

Multiplication of integers

Using mathematical principles	Using physical examples
A $3 + {}^{-}3 =$ ▒ $5 \cdot (3 + {}^{-}3) =$ ▒ $(5 \cdot 3) + (5 \cdot {}^{-}3) =$ ▒ $15 + 5 \cdot {}^{-}3 =$ ▒ This is the number that adds to 15 to give 0. → $5 \cdot {}^{-}3 =$ ▒ ${}^{-}3 \cdot 5 =$ ▒	**C** Negative ← → Positive −15 −10 −5 0 5 10 15 Traveling 5 mph to the **right:** → Think about → 3 hours **from now** → and decide → at what point the car will be. ↓ 5 × 3 = ▒ Car is now at 0. Traveling 5 mph to the **right:** → Think about → 3 hours **ago** → and decide → at what point the car was. ↓ 5 × −3 = ▒ Traveling 5 mph to the **left.** → Think about → 3 hours **ago** → and decide → at what point the car was. ↓ −5 × −3 = ▒
B $5 + {}^{-}5 =$ ▒ ${}^{-}3 \cdot (5 + {}^{-}5) =$ ▒ $({}^{-}3 \cdot 5) + ({}^{-}3 \cdot {}^{-}5) =$ ▒ ${}^{-}15 + {}^{-}3 \cdot {}^{-}5 =$ ▒ This is the number that adds to −15 to give 0. → ${}^{-}3 \cdot {}^{-}5 =$ ▒ ${}^{-}5 \cdot {}^{-}3 =$ ▒	

Discussion Exercises

1. Copy Example A on your paper, correctly filling in each ▒. If you can, for each step give a basic principle that assures you that the statement is true.
2. Copy Example B on your paper, correctly filling in each ▒. If you can, for each step give a basic principle to explain your reasoning.
3. Copy and complete each equation in Example C.

Exercises

1. Find the integer for ***a***. Then find the integer for ***b***.

[a] $^{-}7 \cdot 1 = a \longrightarrow 1 \cdot {}^{-}7 = b$
[b] $^{-}12 \cdot 1 = a \longrightarrow 1 \cdot {}^{-}12 = b$
[c] $5 \cdot 0 = a \longrightarrow 0 \cdot 5 = b$
[d] $^{-}5 \cdot 0 = a \longrightarrow 0 \cdot {}^{-}5 = b$

2. You can use the distributive principle and the one principle to help you find products. Study the examples.

$2 \cdot {}^{-}3 = (1 + 1) \cdot {}^{-}3 = (1 \cdot {}^{-}3) + (1 \cdot {}^{-}3) = {}^{-}3 + {}^{-}3 = {}^{-}6$
$3 \cdot {}^{-}5 = (1 + 1 + 1) \cdot {}^{-}5 = (1 \cdot {}^{-}5) + (1 \cdot {}^{-}5) + (1 \cdot {}^{-}5) = {}^{-}5 + {}^{-}5 + {}^{-}5 = {}^{-}15$

Find the integer for ***a***. Then find the integer for ***b***.

[a] $^{-}3 + {}^{-}3 = a \longrightarrow 2 \cdot {}^{-}3 = b$
[b] $^{-}4 + {}^{-}4 = a \longrightarrow 2 \cdot {}^{-}4 = b$
[c] $^{-}5 + {}^{-}5 = a \longrightarrow 2 \cdot {}^{-}5 = b$
[d] $^{-}7 + {}^{-}7 = a \longrightarrow 2 \cdot {}^{-}7 = b$
[e] $^{-}58 + {}^{-}58 = a \longrightarrow 2 \cdot {}^{-}58 = b$
[f] $^{-}4 + {}^{-}4 + {}^{-}4 = a \longrightarrow 3 \cdot {}^{-}4 = b$
[g] $^{-}5 + {}^{-}5 + {}^{-}5 + {}^{-}5 = a \longrightarrow 4 \cdot {}^{-}5 = b$
[h] $^{-}8 + {}^{-}8 + {}^{-}8 + {}^{-}8 = a \longrightarrow 4 \cdot {}^{-}8 = b$
[i] $^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3 = a \longrightarrow 5 \cdot {}^{-}3 = b$
[j] $^{-}6 + {}^{-}6 + {}^{-}6 + {}^{-}6 + {}^{-}6 = a \longrightarrow 5 \cdot {}^{-}6 = b$

3. Find the integer for ***a***. Then find the integer for ***b***.

[a] $2 \cdot {}^{-}3 = a \longrightarrow {}^{-}3 \cdot 2 = b$
[b] $2 \cdot {}^{-}5 = a \longrightarrow {}^{-}5 \cdot 2 = b$
[c] $2 \cdot {}^{-}47 = a \longrightarrow {}^{-}47 \cdot 2 = b$
[d] $2 \cdot {}^{-}34 = a \longrightarrow {}^{-}34 \cdot 2 = b$
[e] $3 \cdot {}^{-}7 = a \longrightarrow {}^{-}7 \cdot 3 = b$
[f] $4 \cdot {}^{-}6 = a \longrightarrow {}^{-}6 \cdot 4 = b$
[g] $7 \cdot {}^{-}9 = a \longrightarrow {}^{-}9 \cdot 7 = b$
[h] $10 \cdot {}^{-}23 = a \longrightarrow {}^{-}23 \cdot 10 = b$

4. Copy the equations and write the correct number for each ▒. Each equation can be completed by thinking about the equation above it and a basic principle or multiplication fact.

[a]
$2 + {}^{-}2 = ▒$
$3 \cdot (2 + {}^{-}2) = ▒$
$(3 \cdot 2) + (3 \cdot {}^{-}2) = ▒$
$6 + 3 \cdot {}^{-}2 = ▒$
$3 \cdot {}^{-}2 = ▒$

This number adds to 6 to give zero. Therefore, $(3 \cdot {}^{-}2)$ must be the integer

[b]
$2 + {}^{-}2 = ▒$
$^{-}3 \cdot (2 + {}^{-}2) = ▒$
$({}^{-}3 \cdot 2) + ({}^{-}3 \cdot {}^{-}2) = ▒$
$^{-}6 + {}^{-}3 \cdot {}^{-}2 = ▒$
$^{-}3 \cdot {}^{-}2 = ▒$

This number adds to $^{-}6$ to give zero. Therefore, $({}^{-}3 \cdot {}^{-}2)$ must be the integer

[c]
$4 + {}^{-}4 = ▒$
$^{-}5 \cdot (4 + {}^{-}4) = ▒$
$({}^{-}5 \cdot 4) + ({}^{-}5 \cdot {}^{-}4) = ▒$
$^{-}20 + {}^{-}5 \cdot {}^{-}4 = ▒$
$^{-}5 \cdot {}^{-}4 = ▒$

This number adds to $^{-}20$ to give zero. Therefore, $({}^{-}5 \cdot {}^{-}4)$ must be the integer

5. Use the correct word (**positive** or **negative**) to complete each statement.

[a] The product of a positive integer and a negative integer is a __?__ integer.

[b] The product of a negative integer and a negative integer is a __?__ integer.

6. Find the products.

[a] $4 \cdot {}^{-}5 =$ **P**
[b] $3 \cdot {}^{-}5 =$ **P**
[c] $2 \cdot {}^{-}5 =$ **P**
[d] $1 \cdot {}^{-}5 =$ **P**
[e] $0 \cdot {}^{-}5 =$ **P**
[f] $^{-}1 \cdot {}^{-}5 =$ **P**
[g] $^{-}2 \cdot {}^{-}5 =$ **P**
[h] $^{-}3 \cdot {}^{-}5 =$ **P**
[i] $^{-}4 \cdot {}^{-}5 =$ **P**
[j] $^{-}5 \cdot {}^{-}5 =$ **P**
[k] $^{-}6 \cdot {}^{-}5 =$ **P**
[l] $^{-}7 \cdot {}^{-}5 =$ **P**

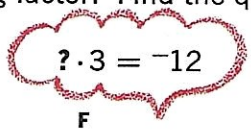
7. Find the products.

[a] $2 \cdot {}^{-}4$
[b] $^{-}4 \cdot 3$
[c] $^{-}4 \cdot {}^{-}3$
[d] $^{-}1 \cdot 1$
[e] $^{-}1 \cdot {}^{-}1$
[f] $^{-}2 \cdot {}^{-}4$
[g] $^{-}8 \cdot 0$
[h] $12 \cdot {}^{-}3$
[i] $^{-}3 \cdot 12$
[j] $^{-}3 \cdot {}^{-}12$
[k] $^{-}4 \cdot 8$
[l] $^{-}4 \cdot {}^{-}8$
[m] $^{-}5 \cdot 0$
[n] $^{-}7 \cdot 6$
[o] $^{-}7 \cdot {}^{-}6$
[p] $^{-}12 \cdot 13$
[q] $^{-}12 \cdot {}^{-}13$
[r] $14 \cdot {}^{-}12$
[s] $13 \cdot {}^{-}13$
[t] $^{-}13 \cdot {}^{-}13$
[u] $10 \cdot {}^{-}27$
[v] $^{-}100 \cdot 38$
[w] $^{-}100 \cdot {}^{-}38$
[x] $468 \cdot {}^{-}1000$

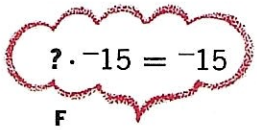
8. Find the missing factors.

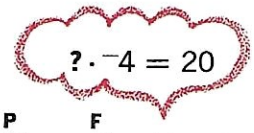
F F P		F F P		F F P
[a] $^{-}2 \cdot a = {}^{-}8$	[f] $r \cdot {}^{-}4 = 0$	[k] $3 \cdot n = {}^{-}18$	[p]	$n \cdot 20 = 80$
[b] $8 \cdot b = {}^{-}16$	[g] $u \cdot 5 = {}^{-}5$	[l] $s \cdot 5 = {}^{-}30$	[q]	$n \cdot {}^{-}20 = 80$
[c] $^{-}3 \cdot x = 3$	[h] $m \cdot 3 = {}^{-}9$	[m] $2 \cdot y = {}^{-}10$	[r]	$n \cdot 20 = {}^{-}80$
[d] $3 \cdot x = 12$	[i] $^{-}3 \cdot y = {}^{-}3$	[n] $r \cdot {}^{-}2 = {}^{-}8$	[s]	$n \cdot 11 = {}^{-}55$
[e] $c \cdot {}^{-}2 = 6$	[j] $t \cdot {}^{-}1 = 8$	[o] $n \cdot {}^{-}3 = 12$	[t]	$^{-}12 \cdot y = 36$

9. You can find the quotient of two integers by thinking about a missing factor. Find the quotients.



P F		P F		P F
[a] $^{-}12 \div 3 =$ **F**	[b] $^{-}15 \div {}^{-}15 =$ **F**	[c]	$20 \div {}^{-}4 =$ **F**
[d] $^{-}35 \div 7 = n$	[e] $12 \div 4 = n$	[f]	$12 \div {}^{-}4 = n$
[g] $^{-}12 \div 4 = n$	[h] $^{-}12 \div {}^{-}4 = n$	[i]	$8 \div {}^{-}8 = n$
[j] $^{-}8 \div {}^{-}8 = n$	[k] $0 \div {}^{-}6 = n$	[l]	$20 \div {}^{-}5 = n$
[m] $28 \div {}^{-}7 = n$	[n] $^{-}28 \div 7 = n$	[o]	$^{-}28 \div {}^{-}7 = n$
[p] $^{-}16 \div {}^{-}1 = n$	[q] $^{-}16 \div 1 = n$	[r]	$^{-}45 \div 15 = n$
[s] $^{-}45 \div {}^{-}15 = n$	[t] $^{-}2448 \div {}^{-}16 = n$	[u]	$[(^{-}8 \cdot 7) \div {}^{-}8] = n$





★**10.** Sue is thinking of a number. If you multiply Sue's number by itself and then add it to that product, you get 20. Give two possibilities for the number Sue might be thinking about.

Think

Ⓐ A year on earth is 365 days.
A year on Mars is $687\frac{1}{2}$ "earth days."

Ⓑ An object on Mars weighs $\frac{2}{5}$ as much as it does on earth.

Ⓒ Assume that a person can jump $\frac{5}{2}$ as high on Mars as he can on earth.

Translate the following earth story into a story that would be true on Mars. (Write Tom's age to the nearest Mars year.)

The surface of Mars as reported by the Mariner IV spacecraft

Today was Tom's birthday. He was 14 years old. He stepped on the scales and found that he weighed 95 pounds. On his way to school Tom jumped over a fence. It was 4 feet high, and Tom could just make it over the top.

13-6 INTEGER INEQUALITIES AND GRAPHING

Jan and Tod played eight games. Here are their point scores for each game. The colored numerals indicate scores "in the hole" or "going set."

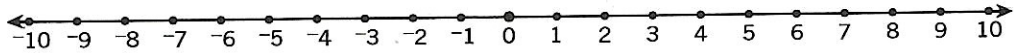
Games	1	2	3	4	5	6	7	8
Jan's scores	7	−5	9	0	−8	−2	6	−5
Tod's scores	−3	2	6	−1	6	−7	8	−10

Discussion Exercises

1. For each game, tell who had the best score.

2. For each game, tell how many points the loser needed in order to tie the winner.

It is important to be able to tell which of two integers is larger. The discussion above and the following number line may help you complete the exercises on inequalities.



Exercises

1. Give the correct sign ($<$ or $>$) for each ●.

[a] Since $3 = 2 + 1$, we know that $3 \bullet 2$.

[b] Since $4 = 3 + 1$, we know that $4 \bullet 3$.

[c] Since $5 = 4 + 1$, we know that $5 \bullet 4$.

[d] Since $-3 = -4 + 1$, we know that $-3 \bullet -4$.

[e] Since $-4 = -5 + 1$, we know that $-4 \bullet -5$.

[f] Since $-5 = -6 + 1$, we know that $-5 \bullet -6$.

[g] Since $-8 + 1 = -7$, we know that $-8 \bullet -7$.

[h] Since $-9 + 1 = -8$, we know that $-9 \bullet -8$.

[i] Since $-1 + 1 = 0$, we know that $-1 \bullet 0$.

[j] Since $-3 > -4$ and $-4 > -5$, we know that $-3 \bullet -5$.

[k] Since $-9 < -8$ and $-8 < -7$, we know that $-9 \bullet -7$.

[l] Since $1 > 0$ and $0 > -2$, we know that $1 \bullet -2$.

[m] Since $8 > 0$ and $0 > -9$, we know that $8 \bullet -9$.

2. Use **greater than** or **less than** to complete each sentence.

[a] Any positive integer is __?__ zero.

[b] Any negative integer is __?__ zero.

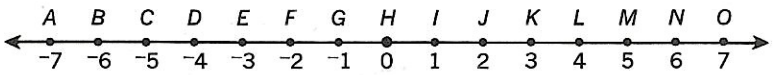
[c] Any positive integer is __?__ any negative integer.

[d] Any negative integer is __?__ any positive integer.

[e] Any integer is __?__ any other integer that is to the left of it on the number line.

[f] Any integer is __?__ any other integer that is to the right of it on the number line.

3. Give the letters for the points in each set.

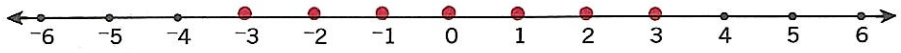


[a] The set of points for numbers greater than -4 and less than zero

[b] The set of points for numbers less than or equal to -2

[c] The set of points for numbers greater than 3 or less than -5

4. The example shows the graph of the set of integers greater than $^{-}4$ and less than 4. We say that we have graphed all possible integers for n in the inequality $^{-}4 < n < 4$.



Draw number lines and graph each set.

[a] Integers greater than 0 and less than 5

[b] Integers greater than $^{-}2$ and less than 3

[c] Integers greater than $^{-}6$ and less than 6

[d] Integers less than 0 and greater than $^{-}6$

[e] Integers less than 1 and greater than $^{-}1$

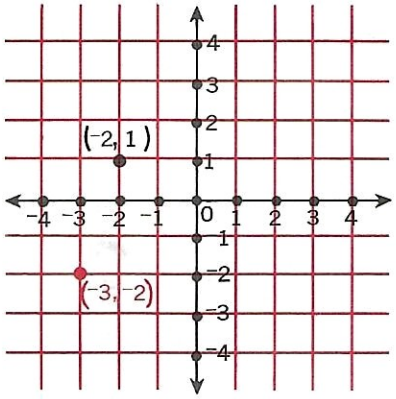
[f] Integers greater than $^{-}8$ and less than $^{-}2$

[g] Integers less than $^{-}5$ and greater than $^{-}9$

★[h] Integers for n in the inequality $^{-}2 < n < 5$

★[i] Integers for n in the inequality $^{-}6 < n < ^{-}1$

5. If we use two number lines, we can show the graph of any pair of integers. The figure shows the graphing of the integer pair $(^{-}3, ^{-}2)$. We call the colored dot **the graph of the integer pair**. The graph of the integer pair $(^{-}2, 1)$ is shown with a heavy black dot. For each exercise, use graph paper and draw two number lines like those shown. Then graph the pairs indicated.



[a] Graph the integer pairs and write the integer pair beside each point.
$(4, 2)$, $(4, ^{-}2)$, $(4, 0)$, $(^{-}3, 4)$, $(^{-}3, ^{-}4)$, $(^{-}3, 0)$, $(^{-}2, 2)$, $(2, ^{-}2)$, $(^{-}2, ^{-}2)$, $(2, 2)$

[b] Graph the set of integer pairs whose second number is $^{-}2$; that is, $(4, ^{-}2)$, $(3, ^{-}2)$, and so on.

[c] Graph the set of integer pairs whose first number is $^{-}3$.

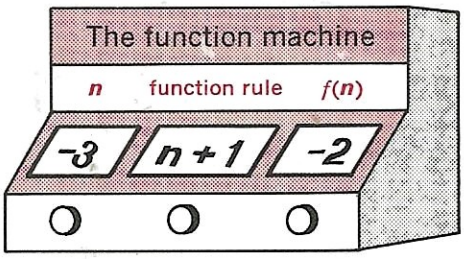
[d] Graph the set of integer pairs in which the first and second number add to give zero.

★**6.** Draw two number lines and then graph the set of integer pairs in which

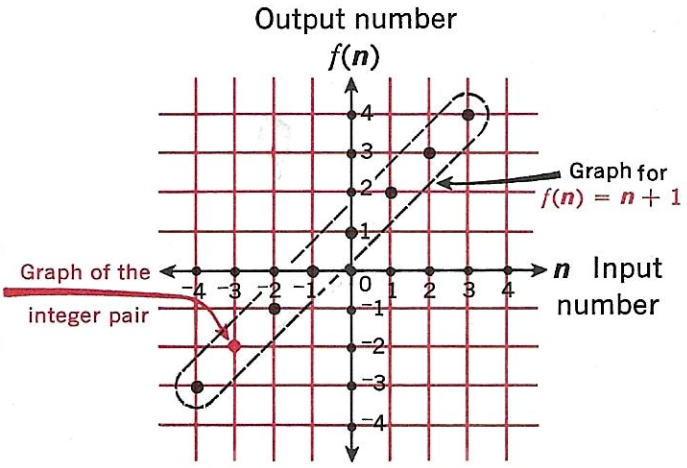
[a] the first number is equal to the second
[b] the first number is greater than the second
[c] the second number is greater than the first
[d] the second number is two times the first number
[e] the second number is the first number squared
[f] twice the second number is less than the first
[g] twice the second number is three times the first

GRAPHING FUNCTIONS

Study the example in order to see how to graph a function by graphing pairs of integers.



function rule $n + 1$		Integer pairs
n	$f(n)$	↓
-4	-3	$\rightarrow (-4, -3)$
-3	-2	$\rightarrow (-3, -2)$
-2	-1	$\rightarrow (-2, -1)$
-1	0	$\rightarrow (-1, 0)$
0	1	$\rightarrow (0, 1)$
1	2	$\rightarrow (1, 2)$
2	3	$\rightarrow (2, 3)$
3	4	$\rightarrow (3, 4)$



Exercises

For each exercise use graph paper and draw two number lines as in the example (page 423). Copy and complete the table. Then graph each number pair. Draw a loop around the points to help you see the pattern of the graph.

1.

function rule	
$n + 2$	
n	$f(n)$
2	4
1	
0	
-1	
-2	
-3	
-4	

2.

function rule	
$3 - n$	
n	$f(n)$
-4	7
-3	
-2	
-1	
0	
1	
2	

3.

function rule	
$-1 \cdot n$	
n	$f(n)$
-4	4
-3	
-2	
-1	
0	
1	
2	

Think

Give the number pair. In each exercise, Part a is done for you.

1.

	Sums	+	×	Products
[a]	0	5	-5	-25
[b]	0			-49
[c]	-8			16
[d]	8			16
[e]	-4			0
[f]	2			-15
[g]	1			-20
[h]	-20			100

2.

	Sums	+	÷	Quotients
[a]	-10	-5	-5	1
[b]	-9			8
[c]	-2			1
[d]	-4			-2
[e]	-20			1
[f]	8			-5
[g]	-40			7
[h]	-8			0

CHAPTER REVIEW

Key Concepts

1. As centuries passed, man began to think carefully about whole numbers, and it became necessary to invent, for each whole number, a number that when added to the whole number gives the sum zero. On the number line, the numbers for the points opposite the whole number points are called **negative** integers. The nonzero whole numbers (positive integers), together with the negative integers and zero, form a set of numbers called **integers**.
2. We can think about addition and multiplication of integers by using physical examples or by using basic principles. The important basic principles are the **opposites, zero, one, commutative, associative,** and **distributive principles**.
3. We can subtract integers easily by thinking about missing addends. We can divide integers by thinking about missing factors.
4. The number line helps us to understand inequalities of integers. Any integer is greater than any other integer that is to the left of it on the number line. We use the symbols $>$ (greater than) and $<$ (less than) to indicate these inequalities.
5. Interesting relationships between integers can be observed by **graphing** integers on the number line and by graphing pairs of integers in the plane.

Vocabulary List

function
graph
integer
integer pair
negative integer
opposite
opposites principle
positive integer

Review Exercises

1. List all the negative integers greater than -10.
2. List all the integers between -5 and 5.
3. Give the opposite of each integer.

   [a] 5 [b] -3 [c] 27 [d] -16 [e] -365 [f] 273 [g] -5806 [h] 0

4. For each exercise, give the basic principle illustrated.

   [a] $-28 \cdot 1 = -28$
   [b] $-7 \cdot 1 = 1 \cdot -7$
   [c] $-12 + 12 = 0$
   [d] $3 \cdot (8 + -8) = (3 \cdot 8) + (3 \cdot -8)$
   [e] $-7 \cdot (5 \cdot -2) = (-7 \cdot 5) \cdot -2$
   [f] $-68 + 0 = -68$
   [g] $-8 + 6 = 6 + -8$

   (A) Opposites
   (B) Zero
   (C) Commutative (+)
   (D) Associative (+)
   (E) One
   (F) Commutative ($\times$)
   (G) Associative ($\times$)
   (H) Distributive

5. Find the sums.

   [a] $9 + -9$ [b] $10 + -8$ [c] $-7 + -5$ [d] $-3 + -8$ [e] $-8 + 5$ [f] $3 + -9$ [g] $-4 + 7$ [h] $-17 + 17$ [i] $-27 + 0$

6. Find the missing addends.

   [a] $0 - -3 = \blacksquare$
   [b] $1 - -3 = \blacksquare$
   [c] $-6 - -4 = \blacksquare$
   [d] $2 - -3 = \blacksquare$
   [e] $8 - -5 = \blacksquare$
   [f] $-6 - 1 = \blacksquare$
   [g] $-8 - 5 = \blacksquare$
   [h] $-2 - -5 = \blacksquare$

7. Find the number for a. Then find the number for b.

   [a] $-9 + -9 = a \longrightarrow 2 \cdot -9 = b$
   [b] $-10 + -10 + -10 = a \longrightarrow 3 \cdot -10 = b$

8. Find the products.

   [a] $9 \cdot 8$
   [b] $-9 \cdot 4$
   [c] $7 \cdot -12$
   [d] $5 \cdot -5$
   [e] $-11 \cdot 11$
   [f] $-8 \cdot -7$
   [g] $-86 \cdot -10$
   [h] $-7 \cdot 20$
   [i] $-1 \cdot -13$
   [j] $-75 \cdot 0$

9. Copy the equations, giving the correct number for each ▒.

[a]
$$3 + {}^{-}3 = \blacksquare$$
$$7 \cdot (3 + {}^{-}3) = \blacksquare$$
$$(7 \cdot 3) + (7 \cdot {}^{-}3) = \blacksquare$$
$$21 + 7 \cdot {}^{-}3 = \blacksquare$$
$$7 \cdot {}^{-}3 = \blacksquare$$

[b]
$$7 + {}^{-}7 = \blacksquare$$
$${}^{-}3 \cdot (7 + {}^{-}7) = \blacksquare$$
$$({}^{-}3 \cdot 7) + ({}^{-}3 \cdot {}^{-}7) = \blacksquare$$
$${}^{-}21 + {}^{-}3 \cdot {}^{-}7 = \blacksquare$$
$${}^{-}3 \cdot {}^{-}7 = \blacksquare$$

[c]
$$9 + {}^{-}9 = \blacksquare$$
$${}^{-}6 \cdot (9 + {}^{-}9) = \blacksquare$$
$$({}^{-}6 \cdot 9) + ({}^{-}6 \cdot {}^{-}9) = \blacksquare$$
$${}^{-}54 + {}^{-}6 \cdot {}^{-}9 = \blacksquare$$
$${}^{-}6 \cdot {}^{-}9 = \blacksquare$$

10. Answer **true** or **false**.

[a] The sum of two negative integers is a positive integer.

[b] The product of a positive integer and a negative integer is a negative integer.

[c] A number is less than a number to the left of it on the number line.

11. Which numbers are not less than $^{-}2$?

[a] $^{-}5$ [b] 0 [c] $^{-}7$ [d] $^{-}1$

12. Which of the following is a true statement?

[a] $^{-}6 \cdot {}^{-}2 = {}^{-}12$ [c] $^{-}6 - {}^{-}2 = {}^{-}4$

[b] $^{-}6 + {}^{-}2 = 8$ [d] $^{-}6 \div {}^{-}2 = {}^{-}3$

13. Find the missing factors.

[a] $^{-}36 \div {}^{-}9 =$ **F** [c] $45 \div {}^{-}9 =$ **F**

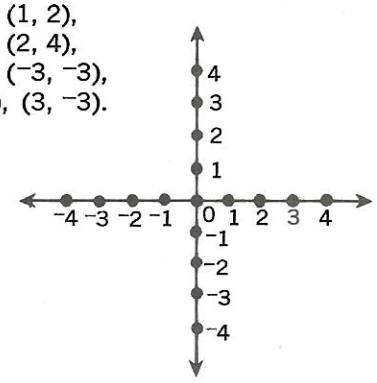
[b] $^{-}28 \div 7 =$ **F** [d] $100 \div {}^{-}10 =$ **F**

14. Which integer is greater, $^{-}99$ or 1?

15. Draw a number line and graph the integers greater than $^{-}4$ and less than 2.

16. Draw two number lines as shown and graph the following pairs. Write the integer pair beside each point.

(3, 1), (1, 2),
(4, 3), (2, 4),
(3, 3), ($^{-}3$, $^{-}3$),
($^{-}3$, 3), (3, $^{-}3$).



CHAPTER SUPPLEMENT

Reading List

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DANTZIG, TOBIAS. *Number—The Language of Science*. Garden City, New York, Doubleday and Company, 1956; pp. 92-93.

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JOHNSON, DONOVAN, and GLENN, WILLIAM. *Adventures in Graphing*. Exploring Mathematics on Your Own; St. Louis, Webster Publishing Company, 1961; pp. 1–30.

MUIR, JANE. *Of Men and Numbers*. New York, Dell Publishing Company, 1961; pp. 259–260.

SCHOOL MATHEMATICS STUDY GROUP. *Mathematics for Junior High School*. New Haven, Yale University Press, 1961; Vol II, Part I, pp. 1–50.

Activities

A. Give four physical examples in which the use of integers is essential and explain these uses to your class.

B. Keep a record of the temperature at a certain time each day for a week. Then, using integers, make a chart showing the changes in temperature for each day.

C. If the opposite of 4 is -4, what do you think would be the opposite of $\frac{3}{4}$? See what ideas you can develop on the theory of negative rational numbers.

D. Using numbers from the set of integers, make up a figure by the use of ordered pairs. Then give your list of ordered pairs to a classmate to graph. Did your classmate come up with the correct drawing?

E. Discuss the concept of closure under addition, subtraction, multiplication, and division for the set of integers.

F. Pick out two or three stocks listed in a daily newspaper and follow the "ups and downs" for a week. Show how the use of integers can help you keep track of average changes in these stocks.

G. Read about the use of relations and functions in graphing. Be ready to report your findings to your class. (See *Adventures in Graphing*.)

TABLES

MATHEMATICAL SYMBOLS

$=$	Is equal to or is the same as	$\cup$	Union
$\neq$	Is not equal to	$\cap$	Intersection
$>$	Is greater than	$\perp$	Is perpendicular to
$<$	Is less than	$\parallel$	Is parallel to
$\leq$	Is less than or equal to	$4!$	Four factorial: $4 \cdot 3 \cdot 2 \cdot 1$
$\approx$	Is approximately	$\sqrt{\ }$	Positive square root of
$\cong$	Is congruent to	$\sqrt[3]{\ }$	Cube root of
$\sim$	Is similar to	$\angle ABC$, $\angle B$	Angle with vertex B
$\{a, b, c\}$	Set of elements a, b, and c	$m\angle ABC$	Measure of angle ABC
$\overline{AB}$	Segment with endpoints A and B	$\overset{\frown}{RS}$	Arc with endpoints R and S
$\ell\overline{AB}$	Length of segment AB	$m\overset{\frown}{RS}$	Measure of arc RS
$\overrightarrow{RS}$	Ray with endpoint R	$35°$	Degree measure of an angle or arc
$\overleftrightarrow{XY}$	Line through points X and Y	$\lvert x \rvert$	Absolute value of x

TABLES OF MEASURES

Prefixes for the metric system

micro-	one millionth (.000001)	deci-	one tenth (.1)	kilo-	one thousand (1000)
milli-	one thousandth (.001)	deca-	ten (10)	mega-	one million (1,000,000)
centi-	one hundredth (.01)	hecto-	one hundred (100)		

Liquid	Capacity	Dry
8 fluid (fl) ounces (oz) = 1 cup 16 fl oz = 2 cups = 1 pint (pt) 32 fl oz = 2 pt = 1 quart (qt) 4 qt = 1 gallon (gal)		2 pt = 1 qt 8 qt = 1 peck (pk) 4 pk = 1 bushel (bu) 1.057 qt = 1 liter (ℓ)

Sound	Speed	Light
Speed of sound: 742 miles per hour (at sea level, 32°F). An object traveling at the speed of sound is traveling at **Mach 1.**		Speed of light: 186,282 miles per second (in a vacuum). The distance light travels in 1 year is called a **light year.** (1 light year = 5,878,496,538,000 miles)

Equivalent units	
1 millimeter (mm) = .03937 inch (in.)* 1 centimeter (cm) = .3937 in. 2.54 cm ≈ 1 in. (≈ means **is approximately.**) 1 decimeter (dm) = 3.937 in. *The yard is defined by law to be $\frac{3600}{3937}$ m.	30.48 cm ≈ 1 foot (ft) .9144 meter (m) ≈ 1 yard (yd) 1 m = 39.37 in. 1 kilometer (km) ≈ .62137 mile (mi) 1.6093 km ≈ 1 mi $\pi \approx 3.1416$

English system	Length	Metric system
12 in. = 1 ft 3 ft = 1 yd $16\frac{1}{2}$ ft = 1 rod (rd) 5280 ft = 1 mi 1760 yd = 1 mi		1000 microns = 1 mm 10 mm = 1 cm 10 cm = 1 dm 10 dm = 1 m 1000 m = 1 km

English system	Weight	Metric system
16 ounces (oz) = 1 pound (lb) 2000 lb = 1 ton		1000 grams (gm) = 1 kilogram (kg) 1000 kg = 1 metric ton

GLOSSARY

abacus A device used for calculating, usually involving sliding beads or counters along a wire.

absolute value The undirected distance from a point on the number line to the origin. The absolute value of a number is never negative. We denote absolute value by vertical lines: $|-5| = 5$; $|5| = 5$; $|0| = 0$.

accuracy The ratio of the possible error of a measurement to the measurement itself. For example, if a measurement is given as 10 in. to the nearest $\frac{1}{2}$ in., then the accuracy is the ratio of $\frac{1}{2}$ to 10, or 5%.

acute angle An angle smaller than a right angle.

addend Any one member of a set of numbers to be added.

addition An operation that combines a first number (**addend**) and a second number (**addend**) to give exactly one number (this result called the **sum** of the two addends).

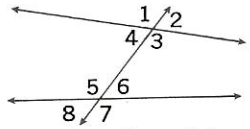
additive inverse Each of two integers whose sum is zero is said to be the additive inverse of the other.

adjacent angles Two angles with a common vertex and a common side. The intersection of the interiors of the angles is the empty set.

adjacent sides Two sides of a polygon with a common endpoint.

algorithm (algorism) A computational procedure, for example, the algorithm for extracting square roots.

alternate exterior angles A transversal of two lines forms eight angles as shown. Angles 1, 2, 7, and 8 are exterior angles. Angles 1 and 7, on alternate sides of the transversal, are a pair of alternate exterior angles, as are angles 2 and 8.



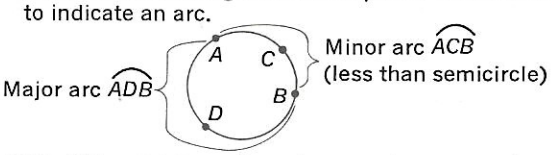
alternate interior angles Two of the four interior angles (3, 4, 5, and 6) shown above which are on alternate sides of the transversal. Angles 3 and 5 form such a pair, as do angles 4 and 6.

altitude (of a triangle) The line segment from any vertex of a triangle perpendicular to the opposite side. The phrase is also used to refer to the perpendicular distance from any vertex to the opposite side.

angle The figure formed by two rays with a common endpoint.

approximation A number is an approximation of another number if the first number is suitably "close" (according to context) to the other number.

arc A part of a circle as indicated in the diagram. Note that one designates three points on a circle to indicate an arc.



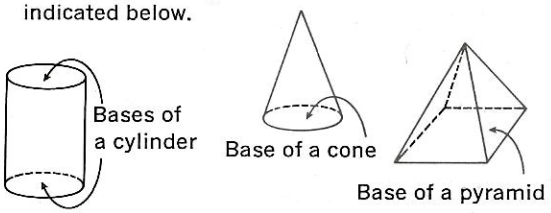
area The measure of a plane region—a number comparing the size of the region to the size of a chosen unit region, usually a square.

arithmetic mean *See* average.

associative principle The principle for addition and multiplication that sums or products of three numbers are the same regardless of grouping: $(a+b)+c=a+(b+c)$, and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

average The average or arithmetic mean of a set of numbers is the quotient of their sum divided by the number of numbers.

base Any side of a triangle may be referred to as a base of the triangle. Bases of some solids are indicated below.

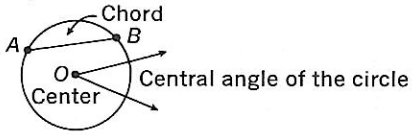


bisect To divide into two congruent parts or into two parts of equal area.

center A point of symmetry for a geometric figure, such as a circle or square. (*See* the figure for chord.)

central angle A central angle of a circle is an angle with its vertex as the center of the circle. (*See* the figure for chord.)

chord A chord of a circle is a line segment with both endpoints on the circle.



circle A set of points, all of which are a specified distance from a given point called the **center.**

circumference The distance around a circle. For each circle, this number is the product of its diameter and the number π (an irrational number approximately $3\frac{1}{7}$, or 3.14).

circumscribed circle If a circle can be drawn through the vertices of a polygon, the circle is said to be circumscribed about the polygon. A circle can be circumscribed about any triangle, but only about certain other polygons.

circumscribed polygon If the sides of a polygon are tangent to a circle, then the polygon is circumscribed about the circle and the circle inscribed in the polygon.

closed figure Intuitively, a figure is closed if you can begin at any point and trace the figure, returning to the initial point.

closure A property of a subset of a set A relative to an operation defined on the set A. For example, the operation of addition on the set of whole numbers is closed relative to the subset of even whole numbers, for the sum of any two evens is even. It is not closed upon the subset of odd whole numbers, for the sum of two odds is not odd. In the set of rational numbers the subset of integers is closed under subtraction, but the subset of whole numbers is not.

collinear (points) Two or more points lying on one line.

common factor An integer that is a factor of each of two or more integers is a common factor of that set of integers. 3 is a common factor of 6 and 9.

common multiple An integer that is a multiple of each of two or more integers is a common multiple of that set of integers. 18 is a common multiple of 2, 3, and 9.

commutative principle The principle for addition and multiplication that sums or products of two numbers are the same regardless of order: $a + b = b + a$, and $a \cdot b = b \cdot a$.

complementary angles Two angles whose sum is a right angle.

complete factorization An expression of a composite number as the product of primes.

composite number A whole number larger than 1 which is not a prime.

concurrent (lines) Two or more lines having only one point in common.

cone The space figure formed by a closed plane curve and all line segments drawn from a point not in the plane of the curve to points of the curve.

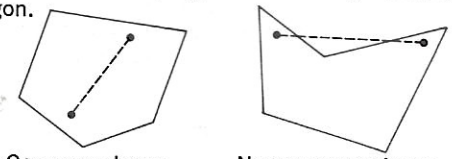
congruent Intuitively, two geometric figures are congruent if they have the same size and shape. Formally, any two sets of points are congruent if the points of one set can be matched one to one with the points of the other so that the distance between any pair of points in one set is the same as the distance between the matching pair in the other set.

congruent angles Intuitively, angles of the same size or degree measure are congruent. Formally, congruent angles are any two angles whose points can be matched one to one in such a way as to preserve all distances.

congruent segments Two segments are congruent if their lengths are equal.

congruent triangles Two triangles are congruent if their sides and angles can be matched so that the members of each matched pair are congruent to each other.

convex polygon A polygon whose interior points all lie on just one side of each side of the polygon. The line segment connecting any two points in the interior of a convex polygon lies entirely within the polygon.

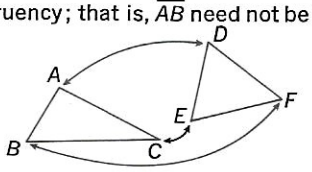


Convex polygon Nonconvex polygon

correspondence A matching of the objects in one set with those of another set. A function.

corresponding angles Two angles matched by any one-to-one correspondence between the angles of two figures.

corresponding points Two points matched by any one-to-one correspondence between the points of two figures. Points A and D, B and F, C and E, are corresponding points. A correspondence need not be a congruency; that is, $\overline{AB}$ need not be congruent to $\overline{DF}$.



cube A rectangular prism all of whose faces are congruent squares.

deca (or deka) A prefix meaning 10 and derived from the Greek.

decagon Any ten-sided polygon.

decimal point The dot that is used in the decimal symbol.

degree The unit angle that is $\frac{1}{90}$ of a right angle.

denominator (of a fraction) For each fraction $\frac{a}{b}$, the whole number b is the denominator of the fraction.

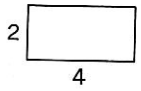
diagonal A line segment connecting two vertices of a polygon which do not lie on the same side.

diameter Any chord of a circle that contains the center is a diameter of the circle.

difference The number resulting from the subtraction operation.

digits The basic symbols used in a place-value system. In our base ten system, the ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

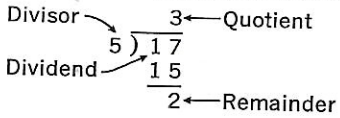
dimensions For a rectangle, the length and the width (as a rectangle of dimensions 4 and 2).



distance The distance between two points is a number. In geometry, we take either the concept of distance or that of congruence as undefined. Then the one concept is defined in terms of the other.

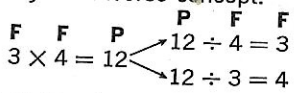
distributive principle The fundamental principle connecting addition and multiplication, permitting us first to find the sum of two addends and then multiply or first to multiply each addend separately and then add: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

dividend The number divided in a division problem:



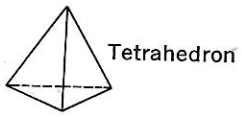
divisible A first whole number is divisible by a second if, upon division by the second, the remainder is zero. If $a = b \cdot c$, and $b \neq 0$, then a is divisible by b.

division Division and multiplication are related to each other by the inverse concept.



divisor *See* dividend.

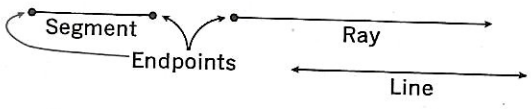
edge A line segment that is a side of any face of a space figure is an edge of the figure. A tetrahedron has six edges, a cube has twelve.



element An element of a set is a member of the set.

empty set The unique set with no elements, symbolized by "{ }" or "$\emptyset$."

endpoint A segment has two, a ray has one, a line has none.



equality The relation of identity. A statement of equality asserts that two symbols name the same object. Because of this definition, we call segments of the same length **congruent segments** rather than equal segments.

equation A sentence using the equals sign, such as $7 = 5 + 2$, is an equation.

equiangular triangle A triangle with all angles of the same measure.

equilateral triangle A triangle with all sides of the same length.

equivalent decimals .5 and .50 are called equivalent decimals, for they represent the same number.

equivalent fractions Fractions which represent the same number, such as $\frac{5}{10}$ and $\frac{50}{100}$.

equivalent ratios Ratios which represent the same rational number.

equivalent sets Sets with the same number of elements.

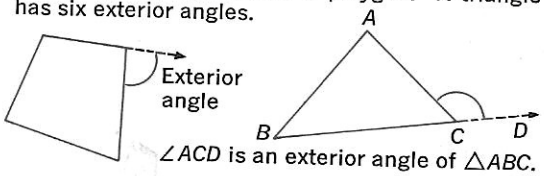
estimate An approximate measure of a set (often used in the sense of a rough calculation).

even number A whole number multiple of 2. Zero is an even number, since $0 = 0 \cdot 2$.

expanded notation A representation of a whole number as a sum of powers of ten, such as: $3425 = \cdot(3 \cdot 10^3) + (4 \cdot 10^2) + (2 \cdot 10^1) + (5 \cdot 10^0)$.

exponent The symbols 4^2 and 5^3 are said to have exponents 2 and 3 respectively. Strictly speaking, each exponent denotes a function. For example, in the symbol 3^2, the 2 denotes the operation of squaring.

exterior angle of a polygon An angle formed by extending any one side of a polygon. A triangle has six exterior angles.



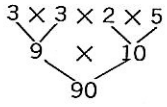
$\angle ACD$ is an exterior angle of $\triangle ABC$.

extremes The first and last terms in a proportion written in the form $a : b = c : d$. The numbers a and d are called the extremes; b and c are called the means. The product of the extremes is equal to that of the means. If $a : b = c : d$, then $a \cdot d = b \cdot c$.

face Any one of the bounding surfaces of a solid. A tetrahedron has four faces; a cube has six.

factor One whole number is a factor of another if it is a divisor of the second: 1, 2, 3, 4, 6, and 12 are all factors of 12.

factor tree A diagram suggestive of a tree (as shown below), presenting the unique prime factorization of a number.



formula A general fact or rule expressed by using symbols. For example, the area of any rectangle of length b and width h is given by $b \cdot h$. Such a formula is usually presented by the equation $A = b \cdot h$, and often the equation is referred to as the formula.

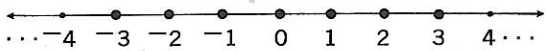
fraction Two definitions are current. Either is logically satisfactory. First, a fraction is a **symbol** of the form $\frac{a}{b}$ where a and b denote integers; $b \neq 0$. Second, a fraction is an ordered pair of integers (a, b) with $b \neq 0$. In algebra, expressions like

$$\frac{3x + 4}{x + 4}$$

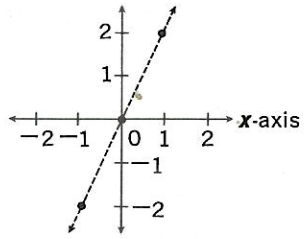
are also called fractions.

function (operation or correspondence) A correspondence that matches each element in a first set with a unique element in a second set. Different elements in the first set may possibly be matched with the same element in the second set.

graph A set of points picturing a relation or function.



Graph of all integers x such that $-3 \leq x \leq 3$



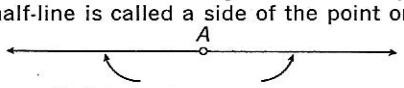
Graph of the function given by $f(x) = 2 \cdot x$

greater than A number x is greater than a number y (written $x > y$) if $x = y + p$ and p is positive. $^{-}3$ is greater than $^{-}5$, for $^{-}3 = {}^{-}5 + 2$. We write, $^{-}3 > {}^{-}5$.

greatest common factor A common factor of a set of numbers which is greater than all other common factors is called the greatest common factor of the set of numbers.

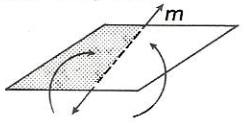
greatest possible error Half of the basic unit in which a measurement is given. For example, if a length is given as $8\frac{3}{4}$ in. to the nearest $\frac{1}{4}$ in., then the basic unit is $\frac{1}{4}$ in., and the greatest possible error is $\frac{1}{8}$ in.

half-line One of the two sets formed by removing any point from a line—a ray minus its endpoint. Each half-line is called a side of the point on the line.



Half-lines formed by point A

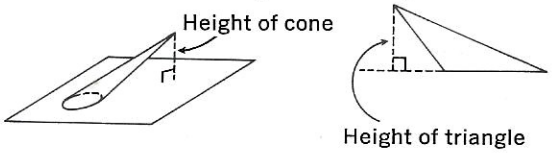
half-plane One of the two sets formed by removing any line from a plane. Each half-plane is called a side of the line in the plane.



Half-planes formed by line m

half-space One of the two regions in space formed by removing any plane from space.

height The height of a triangle from any vertex is the perpendicular distance from that vertex to the line of the opposite side. The height of a tetrahedron from any vertex is the distance to the plane of the opposite face. The height of a prism is the distance between the planes of the parallel faces.



Height of triangle

hexagon A six-sided polygon.

Hindu-Arabic numeral A numeral for a whole number formed from the basic Hindu-Arabic numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

hypotenuse The side of a right triangle opposite the right angle.

identity (element) The number zero is called the identity element for addition since $a + 0 = a$ for any number a. Similarly, the number 1 is the identity element for multiplication since $a \cdot 1 = a$ for any number a.

identity principle The facts that for any number a, $a + 0 = a$ and $a \cdot 1 = a$ are referred to as identity principles (also as the properties of zero and 1).

improper fraction A fraction with numerator greater than or equal to the denominator.

inequality A statement using either of the symbols $>$ or $<$ (greater than or less than), such as $3 < 7$ or $8 + 7 > 4 + 10$.

inscribed angle An angle is said to be inscribed in a circle if its vertex is on the circle and each of its sides contains a chord of the circle.

integers The set consisting of the natural numbers $(1, 2, 3, \ldots)$, the negatives of the natural numbers $(^{-}1, {}^{-}2, \ldots)$, and zero.

interest A charge paid by the borrower for the use of money.

interest rate The percent that indicates the fractional part of the money borrowed that must be paid as interest for each specified unit of time, such as an interest rate of 6% per year, or a rate of 2% per month.

intersection The intersection of two or more sets is the set of all elements common to all sets.

inverse A number and its negative are additive inverses of each other. A number and its reciprocal are multiplicative inverses of each other. Addition and subtraction are referred to as inverse operations, as are multiplication and division.

irrational number A real number not rational, such as π, $\sqrt{3}$, and $\sqrt[3]{7}$.

isosceles triangle A triangle with at least two congruent sides.

least common denominator The smallest common multiple of the denominators of two or more fractions is called the least common denominator for the rational numbers represented by those fractions.

least common multiple The smallest nonzero number that is a multiple of every whole number in a set.

legs (of a right triangle) The perpendicular sides of the right triangle.

length A number designating the measure of a segment (or curve) in terms of a chosen unit is the length of the segment (or curve) in terms of the unit.

less than a is less than b (written $a < b$) if b is greater than a. (*See* greater than.)

line segment A set of points consisting of two points A and B and all points between them. The symbol $\overline{AB}$ is used to denote a segment.

lower terms A phrase used in comparing two equivalent fractions. The fraction in lower terms has the smaller denominator. The fractions $\frac{9}{15}$ and $\frac{18}{30}$ are equivalent and $\frac{9}{15}$ is expressed in lower terms than $\frac{18}{30}$.

lowest terms A fraction that has no lower-term fraction is in lowest terms.

means The second and third terms in a proportion written in the form $a:b = c:d$. (*See* extremes.)

measure Measure is a general term for "length," "area," "volume," and so forth. We speak of the measure of a segment, polygon, solid, or angle.

meter The basic unit of length in the metric system, approximately one ten-millionth of the distance from the north pole to the equator. Our units (inch, foot, etc.) are defined by agreeing that 39.37 in. shall be a meter.

metric geometry The study of properties of geometric figures related to measurement.

metric system The system of weights and measures commonly used in the world, excluding Great Britain and its former colonies.

mixed numeral A numeral like $4\frac{2}{3}$, indicating the sum of a whole number and a rational number: $4\frac{2}{3} = 4 + \frac{2}{3}$.

multiple For whole numbers a, b, and k, if $a = k \cdot b$, then a is a multiple of b. Note that zero is a multiple of every whole number, for $0 = 0 \cdot a$. If a is a multiple of b, then b is a factor of a.

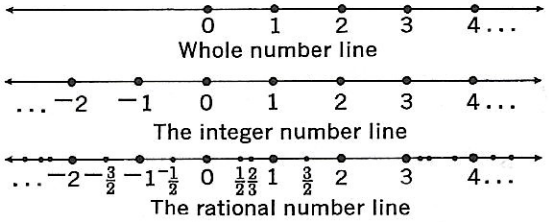
multiplication An operation that combines a first number (**factor**) and a second number (**factor**) to give exactly one number (this result called the **product** of the two factors).

multiplicative inverse (*See* reciprocal.)

negative (*See* opposites principle.)

nonmetric geometry The study of properties of geometric figures without using concepts of measurement.

number line A line with a subset of its points matched with a subset of the real numbers.



number pair An ordered pair of numbers, such as (3, 4). Note that the ordered pair (3, 4) is not the set {3, 4}, for {3, 4} = {4, 3}, while (3, 4) ≠ (4, 3).

numeral A number symbol.

numeration system A set of basic symbols and a procedure for representing numbers with these symbols. Our decimal system uses ten basic symbols to represent whole numbers. An octal system uses eight basic symbols, a binary system uses two, and so forth.

numerator The whole number denoted by the symbol a in any fraction $\frac{a}{b}$.

obtuse angle An angle smaller than a straight angle and greater than a right angle.

odd number A whole number that is not even.

1 principle The identity principle for 1. (*See* identity principle.)

one-to-one correspondence Two sets S and T are in one-to-one correspondence if there is a matching so that each element in S is matched with exactly one element in T. and vice versa.

operation (function or correspondence) A binary operation like "multiplication" associates each ordered pair of numbers with one number. Usually we speak of the four fundamental operations; addition, subtraction, multiplication, and division.

opposites principle The principle that for each integer a, there is an integer b, such that $a + b = 0$. Each integer is called the **opposite** (negative) of the other. The opposite of 4 is $^{-}4$. The opposite of $^{-}\frac{2}{3}$ is $\frac{2}{3}$.

opposite sides In a quadrilateral, any two sides without a common endpoint. In a hexagon, three pairs of sides are designated as pairs of opposite sides.

parallel lines Lines in one plane which do not intersect.

parallelogram A quadrilateral with opposite sides parallel.

pentagon A five-sided polygon.

percent A symbol like 3% is an abbreviation for

$$3 \cdot \tfrac{1}{100}, \text{ or } \tfrac{3}{100}$$

perfect number A number equal to the sum of its proper (less than itself) divisors:

6 is perfect for $6 = 1 + 2 + 3$.
28 is perfect for $28 = 1 + 2 + 4 + 7 + 14$.
496 and 8128 are also perfect.

Only a few more perfect numbers are known, and all those known are even numbers.

perimeter Distance around a closed plane figure.

period When digits are grouped in some regular fashion for some purpose, each group is referred to as a period. In numeration, we group by periods of 3:

421,365,761,409

perpendicular Intersecting lines that form congruent, adjacent angles are perpendicular to each other.

perpendicular bisector A line which both bisects a segment and is perpendicular to it is a perpendicular bisector of the segment.

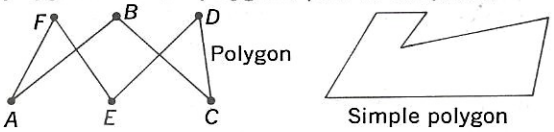
pi (π) The irrational number which is the circumference of a circle whose diameter is 1. The circumference of any circle is $\pi \cdot d$. The area of any circle is $\pi \cdot r^2$ ($\pi \approx \frac{22}{7}$, or 3.1416).

placeholder A numeration system utilizes the concept of place value if each digit in a numeral stands for a multiple of some power of the base number. The particular power is determined by the "place" of the digit in the numeral. In 346, the 3 represents $3 \cdot 10^2$; in 3124, $3 \cdot 10^3$, and so forth. The 3 could be called a placeholder.

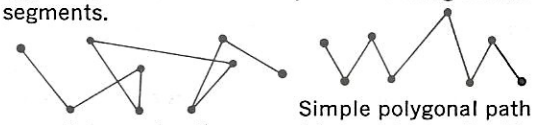
plane figure A set of points in one plane.

point of tangency A point of tangency of a line with a circle is a single point which the line and the circle have in common.

polygon A closed polygonal path in one plane.



polygonal path A connected path consisting of line segments.



polyhedron A three-dimensional figure, each of whose faces is polygonal.

positive integers The set of counting numbers $\{1, 2, 3, 4, \ldots\}$.

postulate An assumption.

power If $a = b^n$, then a is said to be the n^{th} power of b. For example, $1000 = 10^3$, so 1000 is the third power of 10.

precision In comparing measurements, we say that one measurement is more precise than another if its possible error is less. For example, reporting a measurement as $12 \pm \frac{1}{8}$ in. represents a more precise measurement than reporting the same measurement as $12 \pm \frac{1}{2}$ in.

prime factor A prime number which is a factor of another number.

prime factorization *See* complete factorization.

prime number A whole number greater than 1, having only two factors, itself and 1.

prism A three-dimensional figure whose bases are congruent polygons in parallel planes and whose faces are parallelograms.

product The number $a \cdot b$ which results from applying the multiplication operation to the numbers a and b.

proportion An equation stating that two ratios are equal: $a : b = c : d$ or $\frac{a}{b} = \frac{c}{d}$.

protractor An instrument for measuring angles.

pyramid A three-dimensional figure with a polygonal base and triangular lateral faces. A tetrahedron is a triangular pyramid.

quadrilateral A four-sided polygon.

quotient The number $a \div b$ is called the quotient of a and b.

radian A unit of angular measure. A central angle of a circle subtended by an arc whose length is equal to the radius. Approximately 57°; exactly $(180 \div \pi)°$.

radius Any segment from the center of a circle to a point on the circle is a radius of the circle.

ratio The ratio of two numbers is their quotient.

rational number Any number that can be represented by a fraction. Each rational number is the quotient of two integers, the divisor not being zero.

ray A half-line together with the point determining it.

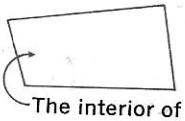
Ray AB or $\overrightarrow{AB}$

reciprocal If $a \cdot b = 1$, each of the numbers a and b is the reciprocal (multiplicative inverse) of the other. Each nonzero real number has a unique reciprocal. Zero has no reciprocal.

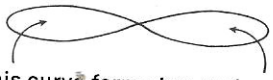
rectangle A parallelogram with four right angles.

reducing to lowest terms Computing the lowest-terms fraction for a rational number.

region A set of points in the plane bounded by a simple closed curve.



The interior of a quadrilateral is a region.



This curve forms two regions.

regrouping Applying the associative principle to regroup addends or factors:

$(3 + 2) + 4 = 3 + (2 + 4)$; $(2 \cdot 3) \cdot 5 = 2 \cdot (3 \cdot 5)$

regular polygon An equiangular and equilateral polygon; for example, an equilateral triangle or a square.

relative error The quotient of the greatest possible error (half the basic measuring unit) divided by the stated measurement. For example, if a measurement is given as 62,000 to the nearest thousand, the relative error is 500 ÷ 62,000, or a little less than 1%.

remainder In whole number division, if dividend D is divided by divisor d and the relation

$$D = (q \cdot d) + r$$

is obtained, where $r < d$, then r is called the remainder. Note that $r = 0$ if d is a factor of D.

repeating decimal A decimal whose digits can be grouped from some point on in repeating periods. Examples: .333. . .; .2727. . .; 4.273149149. . . In these examples, the repeating decimals have (respectively) periods of 1, 2, and 3 digits.

rhombus An equilateral convex quadrilateral which is not a square.

right angle An angle congruent to an adjacent supplementary angle; $\frac{1}{90}$ of a right angle is chosen to be the degree.

Roman numerals The numerals I, V, X, L, C, D, M, and combinations of these numerals, were used by the Romans in their numeration system.

rounding A process of replacing a number by an approximation with fewer significant digits.

scale drawing A drawing of an object made so that distances in the drawing are proportional to actual distances. A scale of 1 to 10 indicates distances in the drawing are $\frac{1}{10}$ of actual distances; a scale of 1000 to 1 might be used to picture a microscopic object.

scalene triangle A triangle with no pair of congruent sides.

scientific notation A notation for writing any number as a power of ten multiplied by some number between 1 and 10 (including 1).

segment *See* line segment.

semicircle Half a circle (one of the two arcs cut off by a diameter).

sequence An ordered array of objects, either finite or infinite. For example, we speak of the sequence of square numbers:

$$1, 4, 9, 16, 25, 36, \ldots$$

first, second, etc.

set The word **set** refers to a basic undefined concept in mathematics. In real life, group and collection are synonyms for set. The English language reserves many special words for certain types of sets, such as hive, flock, herd, covey, pride, tribe, nation.

skew lines Lines not in one plane.

solid figure A three-dimensional closed figure, together with all interior points. The words **cube**, **sphere**, **cone**, and so forth, ordinarily refer to surfaces only. But we may speak of a solid cone or a solid sphere.

space This term usually denotes three-dimensional space. But we also refer to a plane as a two-dimensional space and a line as a one-dimensional space.

sphere The set of all points in space at a fixed distance from some given point.

square A regular quadrilateral.

square root If $x^2 = y$, then x is a square root of y. We use the sign $\sqrt{\ }$ to indicate the operation of extracting the positive square root of a number. For example, in the set of integers, 9 has two square roots, 3 and −3. The symbol $\sqrt{9}$ by agreement denotes only the positive square root, 3.

straightedge An unmarked ruler.

straightedge and compass constructions Constructions possible with straightedge and compass alone. We can only draw lines and circles, utilizing given points. For example, if we are given two points, we can construct the perpendicular bisector of the line segment they determine. We can also construct (given two points) angles of 60°, 30°, 45°, $22\frac{1}{2}°$, 15°, and so forth. However, we cannot construct an angle of 20° if only two points are originally given. This fact settles the famous angle trisection problem which bothered the Grecian geometricans.

subset Set A is a subset of set B if every element of A also belongs to B.

subtend Angles in triangles are said to subtend their opposite sides. Central angles and inscribed angles in a circle are said to subtend the arcs they cut off. Also, sides of a triangle subtend their opposite sides, and chords and arcs in a circle subtend the angles that subtend them.

subtraction Subtraction and addition are related to each other by the inverse concept.

$$\begin{array}{ccc} \text{A} & \text{A} & \text{S} \\ 7 + 8 = 15 & \rightarrow & 15 - 8 = 7 \\ & \rightarrow & 15 - 7 = 8 \end{array} \qquad \begin{array}{c} \text{S} \\ \text{A} \\ \text{A} \end{array}$$

successor If a set is ordered so that there is a first element, second element, third element, and so forth, then the $(n + 1)^{st}$ element is called the successor of the n^{th} element. We speak of **successive elements**.

13, 15, 17 are three successive odd numbers.
81, 100, 121, 144 are four successive squares.

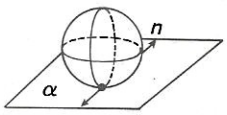
sum A result obtained by adding any set of numbers is referred to as the sum of the numbers. (We also speak, geometrically, of a sum of segments or a sum of angles.)

supplementary angles Two angles (not necessarily adjacent) whose "sum" is a straight angle.

surface area A measure of the set of points on a surface computed by comparing it to a unit region, usually a chosen square region.

tangent (line, plane) A line is tangent to a circle if both are in one plane and have exactly one point in common. A line (or plane) is tangent to a sphere if they have exactly one point in common.

m is tangent to the circle.



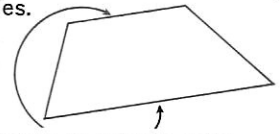
Plane α and line n are tangent to the sphere.

terminating decimal A decimal like .5, 1.24, .0307 and so forth, that represents the quotient of a whole number and a power of ten:

$$.5 = \frac{5}{10}, \qquad 1.24 = \frac{124}{10^2}, \qquad .0307 = \frac{307}{10^4}$$

theorem A statement to be proved true by means of basic assumptions.

trapezoid A quadrilateral with one and only one pair of parallel sides.



Bases, or parallel sides

triangle A three-sided polygon.

union of sets If A and B are sets, then $A \cup B$ (the union of A and B) is the set consisting of all elements that belong to at least one of the two sets.

unit In measurement, we speak of a unit segment, unit square, or unit cube. Other objects are described by comparison with the chosen unit.

variable A letter chosen for convenience in order to refer more easily to properties of a set. For example, we may say: "Every p in the set of natural numbers is either even or odd. If x is odd and y is odd, then $x + y$ is even." Note that one could phrase this second sentence as follows: If a **first number** is odd and a **second number** is odd, then the **sum of the two numbers** is even. Variables are useful since they lend themselves to clarity of expression.

vertex A point is a vertex of a polygon if it lies on two of the sides.

vertical angles Intersecting lines form two pairs of vertical angles.

volume The measure of a set of points in space. This measure compares the set with some chosen unit cube.

whole numbers In elementary mathematics, the set $\{0, 1, 2, \ldots\}$. In advanced mathematics, the phrase usually refers to the set of integers $\{\ldots -2, -1, 0, 1, 2, \ldots\}$.

zero principle The principle that $a + 0 = a$; that is, zero is the identity element for addition. Almost as important as this property of zero is the property that $a \cdot 0 = 0$ for every number a.

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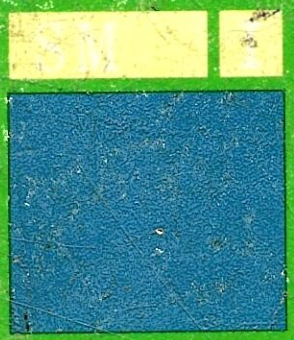

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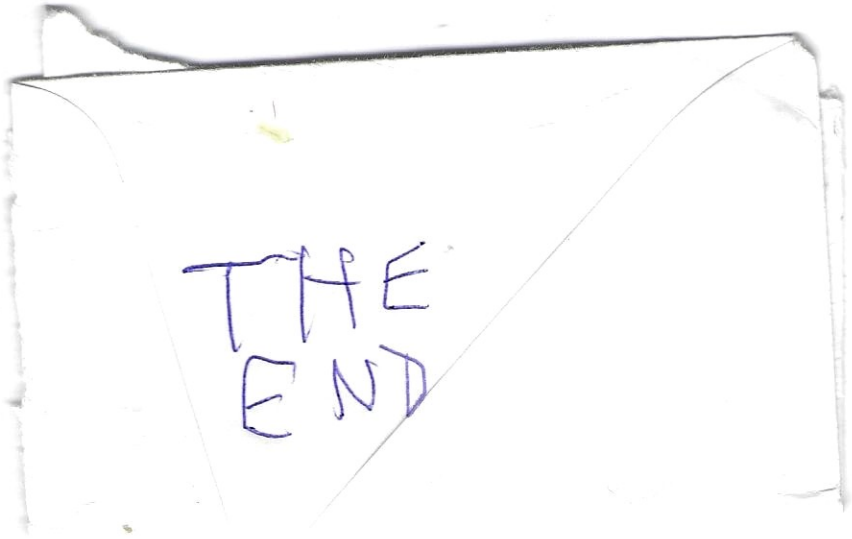
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